A Dynamic Framework for Identification and Estimation of Nonseparable Production Functions

Justin Doty*†

University of Iowa

January 16, 2022

Job Market Paper

[Most Recent Version]

Abstract

This paper studies identification and estimation of a nonseparable model for production functions with unobserved heterogeneity. Nonparametric identification results are established for the production function and productivity process under stationarity conditions. This framework allows for heterogeneous effects of output elasticities and factor efficiencies in addition to nonlinear productivity persistence. It also allows for additional unobservables in the input demand functions, which would violate the scalar unobservability requirement in proxy variables under previous approaches. This extension is used to show firms' heterogeneous responses to productivity shocks corresponding to their productivity history. This paper illustrates these results in an application to U.S. manufacturing firms where the proposed model is estimated using nonlinear quantile regression.

Keywords: Production functions, firm heterogeneity, productivity, uncertainty

JEL Classification: C14, C31, D24, O30

*Department of Economics, University of Iowa, S321 Pappajohn Business Building, 21 E Market St, Iowa City, IA 52242. Email: justin-doty@uiowa.edu

†All replication files for the estimation procedure and interactive plots can be found on the author's personal webpage.

1 Introduction

The production function is a fundamental component of many economic models, and its estimates can be used to study patterns of productivity heterogeneity, returns to scale, and market power. Estimation of the production function is constrained by endogeneity bias from unobserved productivity. The most popular methods for correcting this source of bias impose strict structural assumptions on the functional form of production and restrictions on the number of unobservables in the model. This paper proposes a nonparametric estimation procedure that is robust to these unobservables and captures heterogeneity in firm behavior, which is not found in standard models. My model allows for nonseparability of productivity in the production function and input demand functions. It also allows for nonseparable unobservables beyond the productivity term, which I show to be an important determinant in heterogeneous firm-level estimates. Unlike previous approaches, the structural features of interest are all nonparametrically identified. This contribution is important because the parametric specification used in previous models rely on data that is often not available to researchers.

This paper uses the nonseparability of unobservables to illustrate the importance of capturing the interactions between inputs and productivity. For example, this is used to show that estimates of output elasticities vary over the productivity levels of a firm. It is also used to provide empirical evidence on the non-Hicks neutral effects of productivity, which is shown to vary with respect to firm characteristics, such as their input demand size. Nonseparability of unobservables in the input demand functions are also important and this paper reveals heterogeneous input adjustments with respect to productivity changes. In addition, a flexible productivity evolution process is used to show asymmetric persistence with respect to realizations in productivity shocks at different productivity histories. In contrast, the standard production function approaches assume separability of the unobservables and place more emphasis on addressing the simultaneity bias from unobserved productivity under various timing assumptions on firm input decisions. These proxy variable approaches use a firm's input demand function, which is assumed to be strictly increasing in unobserved productivity. The function is inverted so that productivity can be expressed using observed variables. This is then substituted into the production function, which is estimated in a two-step approach.

This approach was introduced by Olley and Pakes (1996) (hereafter OP) who consider a dynamic optimization problem of a firm who chooses investment to maximize long-run expected profits, and an exit rule which depends on its sell-off value. The investment demand function depends on state variables such as capital stock and unobserved productivity. They show that for positive investment levels this function is invertible in productivity. Their other contribution is correcting a sample selection problem, which is generated by the firm's optimal exit rule. They characterize an equilibrium in which a firm exits the market if their productivity drops below a threshold value determined by its state variables. The selection problem biases estimates of the elasticities corresponding to the state variables. The correction for this is to include the survival probabilities, estimated from a probit regression, as an additional argument in the productivity process for the second stage estimation procedure.

There are two disadvantages to this approach. First, the monotonicity assumption requires discarding observations for which investment is zero. In many plant-level datasets, such as the manufacturing census conducted by Chile, investment levels are often truncated at zero due to high adjustment costs. Second is a violation of the scalar unobservability assumption, which requires that productivity be the only unobservable in the investment demand function. This violation is not unique to their approach and is a common source of identification failure in the proxy variable literature. Intuitively, if there were additional unobservables in the investment demand function, then productivity cannot be expressed as a function of observed variables alone.

Levinsohn and Petrin (2003) (hereafter LP) address the first challenge by providing conditions for which an intermediate input demand function, such as materials, energy, or fuels, is strictly increasing in productivity. This function is used to express productivity as a function of the observed variables. Since many plants report positive use of intermediate inputs, this eliminates the need to discard observations with zero investment levels. Their approach thereafter is similar to OP. They estimate the parameters corresponding to labor in the first stage and state variables in the second stage. An issue with this approach is that if labor is a variable input (chosen to maximize short-run profits), then it is a function of the state variables capital and productivity. This is problematic because productivity is inverted as a function of the same conditioning variables. There exist only specific data-generating processes that can break this functional dependence problem. The paper by Ackerberg *et al.* (2015) (hereafter ACF) provide scenarios in which labor can be identified in the first stage. They propose conditioning on labor in the intermediate input demand function to avoid non-identification of the labor coefficient. This precludes identification in the first stage. Instead, it is included in the second stage with the state variables. This alternative procedure suggests that labor can be chosen prior to or simultaneously as the intermediate inputs. For example, firms will only use certain amounts of material inputs if they know there will be enough workers to utilize them.

The appeal of the control function approaches are its computational simplicity and interpretable timing conditions on input decisions. First stage estimates can be obtained by a polynomial regression and the second stage consists of a nonlinear Generalized Method of Moments (GMM) estimator. The current direction in this literature addresses identification of the model when the input demand functions contain additional unobservables as well as the issue of model specification and its implications for production function estimates.

Invertibility of productivity from the proxy variables is not possible if there are unobserved variables such as demand shocks, input prices, or measurement error. In the OP approach, if the investment demand function contained other unobservables, researchers would not be able to infer values of productivity from different levels of investment. Examples of shocks affecting investment demand include adjustment costs, optimization error, or shocks to product demand. Inversion of multi-dimensional unobservables may be possible if one observes additional proxies, but data on suitable proxies is often not available.[1] The same issue is encountered when intermediate inputs are used as the proxy variable in the LP and ACF framework. Other unobservables, such as measurement error in capital, also poses a serious identification problem since the measurement error appears in both the first and second stage equations nonparametrically. Kim *et al.* (2016) allow for measurement error in capital and other inputs using identification arguments from Hu and Schennach (2008) in the proxy variable framework. Hu *et al.* (2020) (hereafter HHS) take a similar identification approach, but propose an alternative GMM estimator.

Controlling for additional unobservables may reduce some of the unexplained heterogeneity across firms, however there is still a large amount of variation that is left unmodeled. Part of this variation can be accounted for by model specification. The proxy variable approaches typically use a Cobb-Douglas production function with Hicks-neutral productivity. One implication of this specification is that capital shares are assumed constant across firms, which is often rejected by empirical evidence. Some researchers have addressed this by augmenting the parametric specification using firm-specific production functions in a random-coefficient framework.[2] Nonparametric estimation, such as the procedure proposed by Gandhi *et al.* (2020) also show that choice of the production function is important. The proxy variable approach is subject to under-identification due to an instrument-irrelevance problem using a gross-output production function. A value-added model may avoid this critique, however estimates recovered from value-added are fundamentally different from gross-output since

[1] For example, Ackerberg *et al.* (2007) shows that when a demand shock enters the investment function, a firm's pricing decision would be needed to proxy for the additional unobservable and productivity.

[2] See for example Kasahara *et al.* (2017), Balat *et al.* (2018) and Li and Sasaki (2017).

the latter conditions on intermediate inputs. Estimates of TFP and its dispersion ratios will appear more variable using a value-added production function. Their model can be estimated nonparametrically if the productivity term is Hicks-neutral.

The assumption of a Hicks-neutral productivity term implies that technological improvements are not factor-specific. This assumption is difficult to justify empirically, as productivity can be biased towards favoring inputs like labor. Labor-augmenting productivity is an important component to economic models of growth. Therefore, understanding the sources of labor productivity and its heterogeneity, can help explain recent patterns of economic growth, as well as the phenomenon of decreasing labor's share of GDP. Despite its importance, recovering estimates of labor-augmenting productivity is an econometric challenge. To obtain consistent estimates of the production function, the econometrician must be able to correct endogeneity bias with multi-dimensional productivity. Doraszelski and Jaumandreu (2018) suggest an approach that uses the input mix of a firm to invert for factor-augmenting productivity. They use the ratio of material to labor inputs to proxy for the labor-augmenting term, then solve the remaining endogeneity from the Hicks-neutral term by an extension of the proxy variable approach. Their empirical strategy relies on a parametric specification for the production function, so that the decision rules of labor and materials can be expressed as a known function of the data, which include wages, input prices, and output prices. Data at this level is often not available to researchers. It remains to be seen whether similar factor-augmenting estimates can be captured in applications with fewer data requirements while also considering the econometric issues of simultaneity bias and unobservables in the proxy variables.

The identification approach accommodates a productivity process that is nonseparable in innovation shocks. This allows for a current shock to a firm's productivity to change the persistence of previous productivity shocks on future productivity. For example, a firm's history of high productivity may not matter for future productivity if they are hit with a large, negative productivity shock as opposed to a large positive shock. These types of asymmetries are a feature of business cycle fluctuations and a large body of literature have assessed the role of uncertainty in generating these observed patterns. The seminal paper of Bloom (2009) studies the impact of productivity shocks on firms' decisions for hiring and investment. His paper primarily focuses on macro-level shocks and assumes a stochastic volatility process for both macro-level and firm-level business conditions, which are assumed to follow a two-point Markov chain: One state where business conditions are good (expansion) and one where they are bad (recession). Bloom *et al.* (2018) focuses primarily on microeconomic uncertainty and assumes a time-varying process for the volatility in firm productivity. They find that uncertainty in the idiosyncratic productivity process leads to a significant drop in labor and investment, and an increase in the misallocation of labor. These factors contribute to a fall in output driven by the increase in uncertainty. Salgado *et al.* (2019) argue that business cycles are also characterized by changes in third-order moments of productivity (skewness). They show that a change in skewness of firm-level shocks from positive to negative leads to a decrease in investment and output that is more persistent than a change in volatility alone. These papers assume a processes for firm-level shocks that vary over time, but not between firms. The role of asymmetry in productivity histories remains largely unexamined. This paper assumes a productivity process with a general form of conditional heteroskedasticity. Therefore, productivity shocks can exhibit unrestricted volatility and skewness, in addition to changes in its fourth moment (kurtosis) at various points on its conditional distribution.

In this paper, my goal is to provide a framework for identification and estimation of a nonseparable production function and productivity process when there are unobservables in the proxy variables. These dimensions will allow me to examine heterogeneous effects in firm technology, productivity, and input usage. I propose an identification strategy that is an extension of HHS, which uses inputs as instrumental variables (IVs) in the framework of the non-classical measurement error model developed by Hu and Schennach (2008). Their approach uses conditional independence arguments and nonparametric rank conditions to show validity of a proxy variable as an IV for another variable. It is important to note that the identification results of Hu and Schennach (2008) can be applied to nonseparable models, however HHS pursue an alternative strategy by assuming a Cobb-Douglas production function and input demand functions that are additive in unobservables (productivity plus demand shocks). This facilitates less restrictive conditions for identification such as the rank condition, which is difficult to verify in practice. In addition, their model trivially satisfies a normalization assumption on the error term, which for nonseparable models, would require centering a subset of parameters. Their assumptions motivate the construction of a GMM estimator, which relies crucially on the separability of error terms in the model. However, one could question the structural conditions for which the input demand functions are additive in their unobservables. Therefore, a more flexible specification may alleviate these concerns although at the cost of higher-level econometric assumptions. In my paper, these assumptions are needed, however the advantage is that I can consider a richer set of estimates for the production function that has not been captured in previous approaches.

Unlike the GMM estimator proposed by HHS, I propose an estimator that can accommodate nonseparability of the production function in addition to unobservables in the proxy variables. The first extension allows me to capture the non-Hicks neutral effects of productivity. The nonparametric specification I consider does not require a parametric inversion strategy to capture these effects. Since I use inputs as IVs, prices are not needed to invert for productivity. I interpret the interactions between productivity and the inputs as a factor efficiency effect, which is calculated as average derivatives of the production function with respect to inputs and productivity. The second extension allows for heterogeneity in firm input responses with respect to changes in their productivity. For example, firms may have heterogeneous responses in their hiring decisions due to an increase in automation. I also examine how firms adjust their inputs in response to latest changes in their productivity across the entire distribution of input demand. In order to capture the full extent of these heterogeneous responses, I adopt a quantile regression framework using the estimation procedure proposed by Arellano *et al.* (2017), which in turn, is an adaption of Arellano and Bonhomme (2016) and Wei and Carroll (2009) to nonlinear models with time-varying unobservables. This framework allows me to flexibly model conditional distributions, which are used to draw values of productivity from its posterior distribution in a sequential algorithm.

My empirical results show that nonseparability of unobservables are an important determinant in heterogeneous firm-level estimates. I show that estimates of the output elasticities vary with respect to productivity levels and the size of input demands for capital, labor, and materials. For example, I find that capital elasticity exhibits high variation with respect to changes in productivity, which cannot be found in previous production function approaches. For the non-Hicks neutral effects, I find that large firms and firms with high levels of capital intensity tend to be more productive than smaller, less capital intensive firms. I also show firms' heterogeneous input adjustments with respect to productivity changes. For investment and labor adjustments, I find some firms' positive or negative adjustments with respect to productivity levels for different sizes of investment and labor demand. Finally, I show asymmetric impacts of negative and positive innovation shocks at different histories of productivity on future productivity and input demand. For example, I find that low investment firms decrease capital investment faster in response to a large negative productivity shock compared to high investment firms whose adjustments are more gradual.

I introduce the economic model and its restrictions in Section 2. In Section 3, I discuss nonparametric identification. In Section 4 and 5, I discuss estimation based on the econometric restrictions and its implementation. In Section 6, I apply this estimator to U.S. manufacturing firms. Section 7 concludes and provides direction for future research.

2 The Model of Firm Production

In this section, I outline the model for the production function, productivity process, flexible inputs, and investment decisions.

2.1 Production Function with Nonseparable Unobservables

Consider a nonlinear model for a firm's gross-output production function (in logs) given by

$$y_{it} = f_t(k_{it}, l_{it}, m_{it}, \omega_{it}, \eta_{it}), \tag{1}$$

where y_{it} is firm i's output at time t, k_{it} denotes capital, l_{it} denotes labor, which can be flexibly chosen or dynamic, and m_{it} denotes material inputs. The unobserved productivity is denoted by ω_{it}, which is correlated to input choices of the firm. The unobserved production shocks are denoted by η_{it}, which are assumed to be independent of input choices and productivity. The production function, f_t, is assumed to be strictly increasing in η_{it} and can vary over time.

The rank of the unobservable production shock η_{it}, determines the ranking of a firm on the conditional distribution of output. This provides a Skorohod representation of the production function, which is important for developing the econometric restrictions of the model because they are based on conditional quantiles. This representation will also be used for the productivity equation and the input demand functions. Without loss of generality, I re-write the specification for the production function as

$$y_{it} = Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \eta_{it}), \quad \eta_{it}|k_{it}, l_{it}, m_{it}, \omega_{it} \sim Uniform(0, 1), \tag{2}$$

where Q_t^y denotes the conditional quantile function of output. The productivity term enters the production function nonseparably so that interactions between this term and the inputs capture non-Hicks neutral factor efficiency effects. These will be estimated as average derivatives of the production function, which can be interpreted as the increase/decrease in marginal product when there is a small change in productivity levels. I summarize the restrictions on the production function with the following assumptions:

Assumption 2.1 *(Production Function)*

(a) The unanticipated production shocks η_{it} *is independent of* η_{is} *for all* $t \neq s$ *conditional on* $(k_{it}, l_{it}, m_{it}, \omega_{it})$.

(b) The unanticipated production shock η_{it} *follows a standard uniform distribution independent of* $(k_{it}, l_{it}, m_{it}, \omega_{it})$.

(c) $\tau \to Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau)$ *is strictly increasing on* $(0, 1)$.

2.2 Productivity

Productivity, ω_{it}, is assumed to evolve according to a first-order Markov process:

$$\omega_{it} = Q_t^{\omega}(\omega_{it-1}, \xi_{it}), \quad \xi_{it}|\omega_{it-1} \sim Uniform(0, 1), \tag{3}$$

where $\xi_{i1}, \dots, \xi_{iT}$ are independent uniform random variables, which represent innovation shocks to productivity. This specification is not standard in the proxy variable literature. A productivity process which is additive in the innovation shocks are necessary to form conditional moment restrictions in the proxy variable framework. The error term in those models are the differences between realized productivity and the firm's expected productivity. If the true model of productivity is that of Equation (3), then the proxy variable approach would lead to inefficient estimates of the production function. The contribution of nonseparability in the innovation shock is that firm's expectations of future productivity can vary with the size of unanticipated shocks. This specification may better capture the nature of heterogeneous productivity evolution and I use it to show firms' heterogeneous investment and labor responses similar to Bloom *et al.* (2018). It is also likely that firm characteristics play a role in shaping responses to productivity shocks. I consider an extension to endogenous productivity evolution by considering firm knowledge investment from R&D activities similar to Doraszelski and Jaumandreu (2013). To this end, I consider the alternative specification for productivity

$$\omega_{it} = Q_t^{\omega}(\omega_{it-1}, r_{it-1}, \xi_{it}), \quad \xi_{it}|\omega_{it-1}, r_{it-1} \sim Uniform(0, 1), \tag{4}$$

where r_{it-1} denotes R&D expenditures. In this model, ξ_{it} captures the uncertainties in productivity and the R&D process, which I model as

$$r_{it} = Q_t^r(k_{it}, \omega_{it}, \varrho_{it}), \quad \varrho_{it}|k_{it}, \omega_{it} \sim Uniform(0, 1), \tag{5}$$

where ϱ_{it} captures unobserved factors affecting R&D. This extension allows me to examine productivity heterogeneity between firms that perform R&D and those who do not. More specifically, I show that returns to productivity vary between firms subjected to different shocks in the productivity and R&D processes.

Industries with substantial periods of restructuring are also characterized by entry and exit of firms due to changes in future expected productivity levels. Therefore, artificially balancing the data may lead to selection bias if firm's beliefs about future productivity is partially determined by their current productivity. OP show a particular form of bias in the production function estimates in the presence of non-random exit. It is not straightforward to characterize the bias in a nonseparable quantile model, and the tools for correcting selection in these models are still in development. Arellano and Bonhomme (2017) have made significant progress in this regard and propose a selection correction with cross-sectional data. In Appendix D.3, I propose a strategy to correct for non-random exit using their framework. The main contribution of this extension is to show that sample selection may affect the entire distribution of productivity. To summarize the restrictions on productivity, I provide the following assumptions:

Assumption 2.2 *(Productivity)*

(a) The productivity innovation shocks ξ_{it} are independent of ξ_{is} for all $t \neq s$ conditional on ω_{it-1}

(b) ξ_{it} follows a standard uniform distribution independent of previous period productivity ω_{it-1}.

(c) $\tau \rightarrow Q_t^{\omega}(\omega_{it-1}, \tau)$ is strictly increasing on $(0,1)$.

2.3 Flexible Inputs

The firm chooses labor and intermediate inputs to maximize short-term profits. Since I do not restrict the functional form of the production function, it is not necessary to characterize the input decisions as a parametric function of the state variables. Accordingly, I specify the following labor decision rule:

$$l_{it} = Q_t^{\ell}(k_{it}, \omega_{it}, \epsilon_{\ell,it}), \quad \epsilon_{\ell,it}|k_{it}, \omega_{it} \sim Uniform(0,1), \tag{6}$$

where $\epsilon_{\ell,it}$ are i.i.d. and independent of current period state variables. The additional unobservable captures sources of labor demand variation across firms. Using this representation, it is not necessary to describe the distinct sources of heterogeneity across firms; although these can include wages, labor adjustment costs, and other demand shocks to labor. Instead, I interpret it as the ranking index of the firm on the conditional labor distribution. A higher $\tau \in (0,1)$ corresponds to a firm who uses more labor conditional on capital and productivity than a firm with low τ index. With this representation, I can estimate the effects of productivity on labor usage. This is important for understanding how firm's hiring decisions are affected by technological developments such as an increase in automation. I can also consider the case where labor is a dynamic decision variable. This can arise when there are significant hiring/firing costs or industries with high turn-over and employment contracts. A dynamic decision rule for labor can be written as:

$$l_{it} = Q_t^{\ell}(k_{it}, l_{it-1}, \omega_{it}, \epsilon_{\ell,it}), \quad \epsilon_{\ell,it} | k_{it}, l_{it-1}, \omega_{it} \sim Uniform(0,1), \tag{7}$$

where again, $\epsilon_{\ell,it}$ are i.i.d. and independent of current period state variables including previous labor decisions. In Appendix D.2, I show how this model can be used to capture employment decisions in response to adjustment shocks to labor. This is important from a policy perspective for examining unemployment responses to structural changes, which can depend on the magnitude of the shock as well as the size of the firm's labor force.

The firm chooses intermediate inputs to maximize profits. The decision rule is given by:

$$m_{it} = Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \epsilon_{m,it}), \quad \epsilon_{m,it} | k_{it}, l_{it}, \omega_{it} \sim Uniform(0,1), \tag{8}$$

where $\epsilon_{m,it}$ are i.i.d. and independent of current period state variables. I assume material inputs are chosen simultaneously or after labor decisions are made. This is to be consistent with the specification for dynamic labor mentioned earlier. The assumptions on the flexible inputs are summarized as follows:

Assumption 2.3 *(Flexible Inputs)*

(a) The unobserved input demand shocks $\epsilon_{\ell,it}$ and $\epsilon_{m,it}$ are mutually independent over time conditional on (k_{it}, ω_{it}) and $(k_{it}, l_{it}, \omega_{it})$ respectively.

(b) $\epsilon_{\ell,it}$ and $\epsilon_{m,it}$ follow a standard uniform distribution independent of (k_{it}, ω_{it}) and $(k_{it}, l_{it}, \omega_{it})$, respectively.

(c) $\tau \to Q_t^{\ell}(k_{it}, \omega_{it}, \tau)$ and $\tau \to Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \tau)$ are strictly increasing on $(0,1)$.

2.4 Investment

Investment decisions are the solution to a long-run expected profit maximization problem:

$$I_{it} = \iota_t(K_{it}, \omega_{it}) = \underset{I_t \geq 0}{\text{argmax}} \left[\Pi_t(K_{it}, \omega_{it}) - c(I_{it}, \omega_{it}) + \beta \mathbb{E}\big[V_{t+1}(K_{it+1}, \omega_{it+1}) | \mathcal{I}_t\big] \right], \quad (9)$$

where $\Pi_t(\cdot)$ is current period profits as a function of the state variables. Current costs to investment are given by $c(\cdot, \cdot)$, β is the firm's discount factor, and $\mathcal{I}_t$ is the information available to the firm when making investment decisions. I introduce an empirical investment rule (in logs) for (9) given by

$$i_{it} = Q_t^i(k_{it}, \omega_{it}, \zeta_{it}), \quad \zeta_{it} | k_{it}, \omega_{it} \sim Uniform(0, 1). \quad (10)$$

One possible interpretation for ζ_{it} is a shock to investment demand that increases the marginal productivity of capital. In the case where there are many zero observations of investment, I can write a censored version as $i_{it}^* = \max\{0, i_{it}\}$. Although this is not the case in the data considered in this paper, allowing for censoring in investment would be crucial for extending this methodology to other empirical applications. This is easily implemented in my quantile modelling due to the equivariance property of quantiles. Capital accumulates according to the following generalized law of motion

$$K_{it} = \kappa(K_{it-1}, I_{it-1}, \upsilon_{it-1}). \quad (11)$$

Under this specification, capital is determined in period $t-1$. I introduce a random error term, υ_{it-1}, which eliminates the deterministic relationship of the capital accumulation process. This specification is also used by HHS. To summarize the restrictions on the capital process and investment, I assume the following:

Assumption 2.4 *(Capital Accumulation and Investment)*

(a) The unobserved investment demand shocks ζ_{it} is independent of ζ_{is} conditional on (k_{it}, ω_{it}).

(b) ζ_{it} follows a standard uniform distribution independent of (k_{it}, ω_{it}).

(c) The production shock η_{it} and ζ_{it} are independent conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$. In addition, υ_{it} is independent of η_{it} conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$.

(d) $\tau \to Q_t^i(k_{it}, \omega_{it}, \tau)$ *is strictly increasing on* $(0, 1)$.

The next section uses the assumptions on the production function, productivity, flexible inputs, and investment to show that the model is nonparametrically identified. In addition, the assumptions also form econometric restrictions on the model, which I use to estimate firm heterogeneity using nonlinear quantile regressions.

3 Identification

In this section, I show that the conditional densities corresponding to the production function, productivity, input decisions, and investment are nonparametrically identified using Hu and Schennach (2008). To show this, I introduce notation. Let $Z_t = (l_t, k_t, m_t, k_{t+1})$ denote conditioning variables where I have dropped the i subscript for convenience. Assume the following:

Assumption 3.1 *(Conditional Independence):*
$f(y_t|y_{t+1}, I_t, \omega_t, Z_t) = f(y_t|\omega_t, Z_t)$ *and* $f(y_{t+1}|I_t, \omega_t, Z_t) = f(y_{t+1}|\omega_t, Z_t)$.

The first equality of Assumption 3.1 states that conditional on productivity ω_t and Z_t, future output y_{t+1} and current investment I_t do not provide any additional information about current output y_t. The second equality states that conditional on ω_t and Z_t, current investment I_t does not provide any additional information about future output y_{t+1}. These are satisfied by mutual independence assumptions on η_t and ζ_t conditional on $(k_t, l_t, m_t, \omega_t)$ and the fact that η_{it} is assumed to be conditionally independent over time. The next assumption is more technical and requires the following preliminary definition:

Definition 3.1 *(Integral Operator) Let a and b denote random variables with supports $\mathcal{A}$ and $\mathcal{B}$. Given two corresponding spaces $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of functions with domains $\mathcal{A}$ and $\mathcal{B}$, let $L_{b|a}$ denote the operator mapping $g \in \mathcal{G}(\mathcal{A})$ to $L_{b|a}g \in \mathcal{G}(\mathcal{B})$ defined by*

$$[L_{b|a}g](b) \equiv \int_{\mathcal{A}} f_{b|a}(b|a)g(a)da,$$

where $f_{b|a}$ denotes the conditional density of b given a.

With this definition, the uniqueness of an operator mapping can be defined by the next assumption.

Assumption 3.2 *(Injectivity): The operators* $L_{y_t|\omega_t,Z_t}$ *and* $L_{y_{t+1}|\omega_t,Z_t}$ *are injective.*

This allows me to take inverses of the operators. Consider the operator $L_{y_t|\omega_t,Z_t}$. Following Hu and Schennach (2008), injectivity of this operator can be interpreted as its corresponding density $f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)$ having sufficient variation in ω_t given Z_t. This assumption is often phrased as a completeness condition in the nonparametric IV literature on the density $f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)$. More formally, for a given $Z_t \in Supp(Z_t)$,

$$\int f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)g(\omega_t)d\omega_t = 0, \tag{12}$$

for all y_t implies $g(\omega_t) = 0$ for all ω_t. For injectivity of the second operator $L_{y_{t+1}|\omega_t,Z_t}$, one can consider y_{t+1} having sufficient variation for different values of ω_t given Z_t. Since productivity is specified as a Markov process and is highly persistent over time, this assumption is intuitive.

This assumption is more restrictive than that of HHS. Since their model is separable in ω_t, they are able to utilize convolution type arguments, which require conditional independence assumptions and regularity conditions on conditional characteristic functions. I also require two additional assumptions.

Assumption 3.3 *(Uniqueness): For any* $\bar{\omega}_t, \tilde{\omega}_t \in \Omega$*, the set* $\{f_{I_t|\omega_t,Z_t}(I_t|\bar{\omega}_t, Z_t) \neq f_{I_t|\omega_t,Z_t}(I_t|\tilde{\omega}_t, Z_t)\}$ *has positive probability whenever* $\bar{\omega}_t \neq \tilde{\omega}_t$.

This assumption is relatively weak and is satisfied if there is conditional heteroskedasticity in $f_{I_t|\omega_t,Z_t}(I_t|\omega_t, Z_t)$ or if any functional of its distribution is strictly increasing in ω_t. For example, this assumption is satisfied if $E[I_t|\omega_t, Z_t]$ is strictly increasing in ω_t, which is similar to the invertibility conditions required in Olley and Pakes (1996). The flexible accumulation process for capital specified by (11) is necessary for this condition to hold, otherwise investment would be completely determined by k_{t+1} and k_t. In my empirical application, the average investment response to productivity is positive, which supports using the monotonicity restrictions for identification.

Assumption 3.4 *(Normalization): There exists a functional* Γ *such that* $\Gamma[f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)] = \omega_t$.

This functional does not need to be known. It is sufficient to consider a known function of the data distribution as shown by Arellano and Bonhomme (2016). For a nonseparable model, this assumption is satisfied if $E[y_t|\omega_t, Z_t]$ is strictly increasing in ω_t. Then one could normalize

$\omega_t = E[y_t|\omega_t, Z_t]$. In my empirical application, I use a nonseparable Translog production function. In this case, the normalization can be achieved by setting $E[y_t|\omega_t, 0] = \omega_t$, which is standard in the production function with separable productivity. In my model, this requires restrictions on a subset of parameters. With these assumptions, I can now state the first part of the identification results.

Theorem 3.1 *Under Assumptions 3.1, 3.2, 3.3, and 3.4, given the observed density $f_{y_t,I_t|y_{t+1},Z_t}$, the equation*

$$f_{y_t,I_t|y_{t+1},Z_t}(y_t, I_t|y_{t+1}, Z_t) = \int f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t) f_{I_t|\omega_t,Z_t}(I_t|\omega_t, Z_t) f_{\omega_t|y_{t+1},Z_t}(\omega_t|y_{t+1}, Z_t) d\omega_t \tag{13}$$

admits a unique solution for $f_{y_t|\omega_t,Z_t}$, $f_{I_t|\omega_t,Z_t}$, and $f_{\omega_t|y_{t+1},Z_t}$.

Proof: See Appendix B.

This result identifies the conditional density of output and investment. It also identifies the marginal distribution for productivity and the input decision rules as shown in Appendix B. Additional assumptions are needed to identify the Markov transition function for productivity, $f_{\omega_{t+1}|\omega_t}(\omega_{t+1}|\omega_t)$. The requirements for identification of this density are different under two cases involving stationarity and non-stationarity of the density $f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)$.

Corollary 3.1 *(Stationarity): Suppose that the production function is stationary i.e. $f_{y_t|\omega_t,Z_t} = f_{y_1|\omega_1,Z_1}, \forall t \in \{1, \cdots, T\}$. Then, under Assumptions 3.1, 3.2, 3.3, and 3.4, the observed density, $f_{y_t,I_t|y_{t+1},Z_t}$, uniquely determines the density $f_{\omega_{t+1}|\omega_t}$ for any $t \in \{1, \ldots, T-1\}$.*

Proof: See Appendix B.

Corollary 3.2 *(Non-Stationary): Under Assumptions 3.1, 3.2, 3.3, and 3.4, the observed density, $f_{y_{t+1},I_{t+1}|y_{t+2},Z_{t+1}}$, uniquely determines the density $f_{\omega_{t+1}|\omega_t}$ for any $t \in \{1, \ldots, T-2\}$.*

Proof: See Appendix B.

The main conclusion of these two corollaries is that under the condition of stationarity, the productivity process can be identified with $T = 2$ observations per firms, whereas under non-stationarity, the productivity process is identified with $T = 3$ observations per firm. The number of time periods required for identification increases with the length of the auto-regressive process. These data requirements are similar to the control function approach, where the instrument set often includes secondary lags of inputs.

4 Econometric Procedure

This section presents the model specifications and econometric strategy that are used in the empirical application. I consider the following functional form for the production function:

$$Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = \sum_{r_k=0}^{R_k} \sum_{r_l=0}^{R_l} \sum_{r_m=0}^{R_m} \sum_{r_\omega=0}^{R_\omega} \beta_{r_k,r_l,r_m,r_\omega}(\tau) k_{it}^{r_k} l_{it}^{r_l} m_{it}^{r_m} \omega_{it}^{r_\omega}. \tag{14}$$

The above equation is an approximation of the production function in Equation (2) using a truncated product of linear sieves with polynomial basis functions. This specification is similar to the model estimated by Ackerberg and Hahn (2015), although in their model the only unobservable in the production function is productivity, ω_{it}. In their paper, they report the marginal effects of Hicks-neutral productivity on production function elasticities. However, these estimates do not take into account the nature of heterogeneous productivity responses to input composition and scale of the firm. To show how the specification in Equation (14) captures these important effects, I decompose the model into a sum of two series:

$$\begin{aligned} Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = & \sum_{s_k=0}^{S_k} \sum_{s_l=0}^{S_l} \sum_{s_m=0}^{S_m} \gamma_{s_k,s_l,s_m,s_\omega}(\tau) k_{it}^{s_k} l_{it}^{s_l} m_{it}^{s_m} \\ & + \sum_{p_k=0}^{P_k} \sum_{p_l=0}^{P_l} \sum_{p_m=0}^{P_m} \sum_{p_\omega=1}^{P_\omega} \sigma_{p_k,p_l,p_m,p_\omega}(\tau) k_{it}^{p_k} l_{it}^{p_l} m_{it}^{p_m} \omega_{it}^{p_\omega}. \end{aligned} \tag{15}$$

A similar specification was considered by Navarro and Rivers (2018), who extends the identification results of Gandhi *et al.* (2020) to nonseparable production functions. The differences between their model and mine is the nonseparability of the η_{it} term and the input composition effects of materials on productivity, such as a firm's capital-materials or labor-materials ratio. To show how my model can capture these effects, I specify the number of terms in the series as $S_k = S_l = S_m = 2$, $P_k = P_l = P_m = 2$, and $P_\omega = 1$, which corresponds to a Translog production function with first-order interactions of productivity. Accordingly, there are $J_y = 20$ parameters in the production function. Equation (15) can be re-written as:

$$
\begin{aligned}
Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = & \sum_{s_k=0}^{S_k=2}\sum_{s_l=0}^{S_l=2}\sum_{s_m=0}^{S_m=2} \gamma_{s_k,s_l,s_m,s_\omega}(\tau) k_{it}^{s_k} l_{it}^{s_l} m_{it}^{s_m} \\
& + \omega_{it}\Bigg[\sigma_\omega(\tau) + \frac{(\sigma_k(\tau)+\sigma_l(\tau)+\sigma_m(\tau))s_{it}}{3} + \frac{(\sigma_{kk}(\tau)+\sigma_{ll}(\tau)+\sigma_{mm}(\tau))s_{it}^2}{3} \\
& + \frac{(\sigma_k(\tau)-\sigma_l(\tau))(k_{it}-l_{it})}{3} + \frac{(\sigma_{kk}(\tau)-3/2\sigma_{kl}(\tau)+\sigma_{ll}(\tau))(k_{it}-l_{it})^2}{3} \\
& + \frac{(\sigma_k(\tau)-\sigma_m(\tau))(k_{it}-m_{it})}{3} + \frac{(\sigma_{kk}(\tau)-3/2\sigma_{km}(\tau)+\sigma_{mm}(\tau))(k_{it}-m_{it})^2}{3} \\
& + \frac{(\sigma_l(\tau)-\sigma_m(\tau))(l_{it}-m_{it})}{3} + \frac{(\sigma_{ll}(\tau)-3/2\sigma_{lm}(\tau)+\sigma_{mm}(\tau))(l_{it}-m_{it})^2}{3}\Bigg],
\end{aligned}
\tag{16}
$$

where $s_{it} = k_{it} + l_{it} + m_{it}$ denotes scale of the firm. The first line of Equation 16 captures the primary effects of inputs in production. The second to fifth line capture the effects of productivity driven by a Hicks-neutral effect, scale and the input compositions of capital to labor, capital to materials, and labor to materials. The coefficient $\sigma_\omega(\tau)$ captures a Hicks-neutral effect for each rank of the unobservable η_{it}. The identification conditions in the previous section require that, on average, an increase in productivity leads to a proportionate increase in output $\mathbb{E}[\sigma_\omega(\eta_{it})] = \int_0^1 \sigma_\omega(\tau)d\tau = 1$, which corresponds to the standard case in the production function literature. I calculate output elasticities of inputs as *individual* quantile marginal effects, which can vary over the conditional distribution of output and the distribution of input demand. To simplify notation, I let $\beta = (\gamma, \sigma)$. Consider the quantile marginal effect of capital, which I calculate using

$$
\beta_k(\tau_\eta, \tau_k) = \mathbb{E}\left[\frac{\partial Q^y(Q^k(k_{it};\tau_k), l_{it}, m_{it}, \omega_{it}; \beta(\tau_\eta))}{\partial k_{it}}\right], \tag{17}
$$

where τ_η denotes the rank of the conditional output distribution and τ_k denotes the rank of the unconditional capital distribution. This effect is calculated by averaging over ω_{it}, as well as l_{it} and m_{it} evaluated at the fixed percentiles of capital. To capture the variation of the input elasticities with respect to changes in productivity, the marginal effect of capital can be calculated as

$$
\sigma_k(\tau_\eta, \tau_k) = \mathbb{E}\left[\frac{\partial^2 Q^y(Q^k(k_{it};\tau_k), l_{it}, m_{it}, \omega_{it}; \beta(\tau_\eta))}{\partial k_{it}\partial \omega_{it}}\right], \tag{18}
$$

which would provide an interesting comparison to the estimates from Ackerberg and Hahn (2015) and the input composition effects in this paper. For example, the marginal effect of an increase in capital intensity (capital-labor ratio) is derived from Equation (16):

$$\sigma_{kl}(\tau_\eta, \tau_{kl}) = \frac{(\sigma_k(\tau) - \sigma_l(\tau))}{3} + \frac{2(\sigma_{kk}(\tau) - 3/2\sigma_{kl}(\tau) + \sigma_{ll}(\tau))(Q^{kl}(k_{it} - l_{it}; \tau_{kl}))}{3}, \quad (19)$$

where the capital intensity effect is evaluated at percentiles of τ_η and the rank of intensity denoted by τ_{kl}. These effects are interpreted as the effect of an increase in capital intensity on marginal productivity while holding scale of the firm constant.

The Markov process for productivity is specified as a polynomial of degree J_ω:

$$Q_t^\omega(\omega_{it-1}, \tau) = \sum_{j=0}^{J_\omega} \rho_j(\tau)\omega_{it-1}^j. \quad (20)$$

This allows me to capture heterogeneous persistence of productivity, which can depend on the level of previous productivity and the size of the innovation shock. In Appendix D.3, I show that Equation (20) must be modified to account for the fact that unobserved selection alters the productivity distribution rankings. When I augment the productivity model with R&D activities in Appendix D.1, I consider the following specification, which is similar to Doraszelski and Jaumandreu (2013)

$$Q_t^\omega(\omega_{it-1}, r_{it-1}, \tau) = \mathbb{1}\{R_{it-1} = 0\}Q_t^\omega(\omega_{it-1}, \tau) + \mathbb{1}\{R_{it-1} > 0\}Q_t^{\omega,r}(\omega_{it-1}, r_{it-1}, \tau). \quad (21)$$

This allows a firm to adopt corner solutions to R&D expenditure represented by the different functions corresponding to zero or positive R&D. The quantile function $Q^{\omega,r}(\omega_{it-1}, r_{it-1}, \tau)$ can be expressed as a nonlinear function of ω_{it-1} and r_{it-1}. I specify an initial condition for productivity as another polynomial of degree J_{ω_1}.

$$Q^{\omega_1}(k_{i1}, \tau) = \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau)k_{i1}^j. \quad (22)$$

I approximate the input demand functions using tensor product Hermite polynomials in the state variables. For example, I specify the labor input demand function as:

$$Q_t^\ell(k_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_\ell} \alpha_{\ell,j}(\tau)\phi_{\ell,j}(k_{it}, \omega_{it}), \quad (23)$$

where $\phi_{\ell,j}$ is specified as a Hermite polynomial of (k_{it}, ω_{it}). I specify the material input demand function as:

$$Q_t^m(k_{it}, l_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_m} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it}), \tag{24}$$

where $\phi_{m,j}$ is another Hermite polynomial. I specify the investment demand equation as:

$$i_{it} = Q_t^i(k_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_\iota} \delta_j(\tau)\phi_{i,j}(k_{it}, \omega_{it}). \tag{25}$$

In some applications, censoring of investment is problematic. Due to the quantile specification and the equivariance property of quantiles, a censored version of (25) can be adopted. The censored quantile regression model avoids distributional assumptions in estimation at the cost of computational complexity.

It is important to note that the functional forms I consider do not guarantee monotonicity in τ, but the estimator discussed in the next section automatically re-arranges quantiles to enforce monotonicity similar to Chernozhukov *et al.* (2010). It would be interesting to consider other shape restrictions in the quantile modelling presented here. The Translog production function provides a preliminary illustration of the flexibility of the identification and estimation strategy, which can be modified to include returns to scale restrictions. Other shape restrictions may be pursued similarly as Blundell *et al.* (2017), who provide a quantile regression framework for nonseparable demand functions with shape constraints. This approach would provide more structure on the functional forms in this section and is left for future research agenda.

5 Implementation

The following conditional moment restrictions hold as an implication of Assumptions 2.1-2.4 (constant omitted in conditioning set). For the production function:

$$\mathbb{E}\left[\Psi_\tau\left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau))\right)\middle| k_{it}, l_{it}, m_{it}, \omega_{it}\right] = 0. \tag{26}$$

For the labor input demand function:

$$\mathbb{E}\left[\Psi_\tau\left(l_{it} - \sum_{j=0}^{J_\ell} \alpha_{\ell,j}(\tau)\phi_{\ell,j}(k_{it}, \omega_{it})\right)\middle| k_{it}, \omega_{it}\right] = 0. \tag{27}$$

For the material input demand function:

$$\mathbb{E}\left[\Psi_\tau\left(m_{it} - \sum_{j=0}^{J_m} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it})\right)\middle| k_{it}, l_{it}, \omega_{it}\right] = 0. \tag{28}$$

For the investment demand function:

$$\mathbb{E}\left[\Psi_\tau\left(i_{it} - \sum_{j=0}^{J_\iota} \delta_j(\tau)\phi_{\iota,j}(k_{it}, \omega_{it})\right)\middle| k_{it}, \omega_{it}\right] = 0. \tag{29}$$

For the productivity process at $t \geq 2$:

$$\mathbb{E}\left[\Psi_\tau\left(\omega_{it} - \sum_{j=0}^{J_\omega} \rho_j(\tau)\omega_{it-1}^j\right)\middle| \omega_{it-1}\right] = 0, \tag{30}$$

and for initial productivity:

$$\mathbb{E}\left[\Psi_\tau\left(\omega_{i1} - \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau)k_{i1}^j)\right)\middle| k_{i1}\right] = 0, \tag{31}$$

where $\Psi_\tau(u) = \tau - \mathbb{1}\{u < 0\}$. Estimating the parameters from the conditional moment restrictions is infeasible due to the unobserved productivity component. Therefore, I use the following unconditional moment restrictions and posterior distributions for ω_{it} to integrate out productivity. To fix ideas, consider the following unconditional moment restriction corresponding to the production function from Equation (26):

$$\mathbb{E}\left[\int \Psi_\tau\left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau))\right) \otimes \begin{pmatrix} k_{it} \\ l_{it} \\ m_{it} \\ \omega_{it} \end{pmatrix} g_i(\omega_i^T; \theta(\cdot))d\omega_i^T\right] = 0, \tag{32}$$

where $\omega_i^T = (\omega_{i1}, \ldots, \omega_{iT})$ and $\theta(\cdot) = (\beta(\cdot), \alpha_l(\cdot), \alpha_m(\cdot), \delta(\cdot), \rho(\cdot), \rho_{\omega_1}(\cdot))$ denotes a vector of all the model parameters. The posterior density $g_i(\omega_i^T; \theta(\cdot)) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T; \theta(\cdot))$

conditions on the entire set of model parameters. This is due to the equivalence between the density of a random variable and the inverse of the derivative of its quantile function. Therefore, it is impossible to estimate the model parameters in a τ-by-τ procedure. To eliminate the intractability of this problem, the continuous model parameters are approximated by spline functions following Arellano and Bonhomme (2016) and Wei and Carroll (2009). For example, the function $\beta(\tau)$ is approximated by a piecewise-linear interpolating spline on a grid $[\tau_1, \tau_2], [\tau_3, \tau_4], \ldots, [\tau_{Q-1}, \tau_Q]$, contained in the unit interval and constant on $(0, \tau_1]$ and $(\tau_Q, 1)$. Therefore, I write for all $q = 1, \ldots, Q-1$:

$$\beta(\tau) = \beta(\tau_q) + \frac{\tau - \tau_q}{\tau_{q+1} - \tau_q}\big(\beta(\tau_{q+1}) - \beta(\tau_q)\big), \qquad \tau_q < \tau \leq \tau_{q+1}.$$

The intercept coefficient, $\beta_0(\tau)$, is specified as the quantile of an exponential distribution on $(0, \tau_1]$ (indexed by λ_β^-) and $(\tau_{Q-1}, 1)$ (indexed by λ_β^+) given by:

$$\beta_0(\tau) = \beta_0(\tau_1) + \frac{\ln(\tau/\tau_1)}{\lambda_\beta^-}, \quad \tau \leq \tau_1,$$

and

$$\beta_0(\tau) = \beta_0(\tau_Q) + \frac{\ln(1 - \tau/1 - \tau_Q)}{\lambda_\beta^+}, \quad \tau > \tau_Q.$$

The remaining functional parameters are modeled similarly. The usefulness of the piece-wise linear spline is that the posterior density has a closed form expression and does not rely on strong distributional assumptions. For example, the density corresponding to the production function can be written as:

$$\begin{aligned} f_{y_t|k_t,l_t,m_t,\omega_t}(y_t|k_t, l_t, m_t, \omega_t; \beta) = &\sum_{q=1}^{Q-1} \frac{\tau_{q+1} - \tau_q}{Q_t^y(\cdot; \beta(\tau_{q+1})) - Q_t^y(\cdot; \beta(\tau_q))} \mathbb{1}\{Q_t^y(\cdot; \beta(\tau_q)) < y_t \leq Q_t^y(\cdot; \beta(\tau_{q+1}))\} \\ &+ \tau_1 \lambda_\beta^- \exp\big(\lambda_\beta^-(y_t - Q_t^y(\cdot; \beta(\tau_1)))\big) \mathbb{1}\{y_t \leq Q_t^y(\cdot; \beta(\tau_1))\} \\ &+ (1 - \tau_Q)\lambda_\beta^+ \exp\big(-\lambda_\beta^+(y_t - Q_t^y(\cdot; \beta(\tau_Q)))\big) \mathbb{1}\{y_t > Q_t^y(\cdot; \beta(\tau_Q))\}. \end{aligned}$$

The exponential parameters are updated using a likelihood approach:

$$\lambda_\beta^- = \frac{-\mathbb{E}[\int \mathbb{1}\{y_t \leq Q_t^y(\cdot; \beta(\tau_1))\} g_i(\omega_i^T; \theta(\cdot)) d\omega_t]}{\mathbb{E}[\int (y_t - Q_t^y(\cdot; \beta(\tau_1)))\mathbb{1}\{y_t \leq Q_t^y(\cdot; \beta(\tau_1))\} g_i(\omega_i^T; \theta(\cdot)) d\omega_t]},$$

and

$$\lambda_\beta^+ = \frac{\mathbb{E}[\int \mathbb{1}\{y_t > Q_t^y(\cdot; \beta(\tau_Q))\} g_i(\omega_i^T; \theta(\cdot)) d\omega_t]}{\mathbb{E}[\int (y_t - Q_t^y(\cdot; \beta(\tau_Q)))\mathbb{1}\{y_t > Q_t^y(\cdot; \beta(\tau_Q))\} g_i(\omega_i^T; \theta(\cdot)) d\omega_t]}.$$

To estimate the model, the integral inside the expectation of Equation (32) needs to be approximated. This can be done using quadrature methods or Monte Carlo integration and converting the problem into a weighted quantile regression. Due to the high-dimensionality of my application, I choose to use a random-walk Metropolis Hastings algorithm to compute the integral. This becomes a Monte Carlo Expectation Maximization (MCEM) procedure, where the maximization step is performed using quantile regression. The algorithm proceeds as follows. Given an initial parameter value $\hat{\theta}^0$, iterate on $s = 0, 1, 2, \ldots,$ in the following two-step procedure until convergence to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$
\begin{aligned}
g_i(\omega_i^T; \hat{\theta}^{(s)}) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T; \hat{\theta}^{(s)}) \propto \\
\prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \hat{\beta}^{(s)}) f(l_{it}|k_{it}, \omega_{it}; \hat{\alpha_l}^{(s)}) f(m_{it}|k_{it}, l_{it}, \omega_{it}; \hat{\alpha}_m^{(s)}) \\
\times f(i_{it}|k_{it}, \omega_{it}; \hat{\delta}^{(s)}) \prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1}; \hat{\rho}^{(s)}) f(\omega_{i1}|k_{i1}; \hat{\rho}_{\omega_1}^{(s)}).
\end{aligned}
$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve

$$
\begin{aligned}
\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\beta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}^{(m)}; \beta(\tau_q)) \right), \\
\hat{\boldsymbol{\alpha}}_l(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\alpha}_\ell(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(l_{it} - \sum_{j=0}^{J_\ell} \alpha_{\ell,j}(\tau_q) \phi_{l,j}(k_{it}, \omega_{it}^{(m)}) \right), \\
\hat{\boldsymbol{\alpha}}_m(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\alpha}_m(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(m_{it} - \sum_{j=0}^{J_m} \alpha_{m,j}(\tau_q) \phi_{m,j}(k_{it}, l_{it}, \omega_{it}^{(m)}) \right), \\
\hat{\boldsymbol{\delta}}(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\delta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(i_{it} - \sum_{j=0}^{J_\iota} \delta_j(\tau_q) \phi_{\iota,j}(k_{it}, \omega_{it}^{(m)}) \right), \\
\hat{\boldsymbol{\rho}}(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\rho}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(\omega_{it}^{(m)} - \sum_{j=0}^{J_\omega} \rho_j(\tau_q) \omega_{it-1}^{(m)j} \right), \\
\hat{\boldsymbol{\rho}}_{\omega_1}(\tau_q)^{(s+1)} &= \underset{\boldsymbol{\rho}_{\omega_1}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{m=1}^{M} \psi_{\tau_q} \left(\omega_{i1}^{(m)} - \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau_q) k_{i1}^j \right),
\end{aligned}
$$

where $\psi_\tau(u) = (\tau - \mathbb{1}\{u < 0\})u$ is the "check" function from quantile regression. The exponential parameters for the intercept coefficients (e.g. the production function) are updated from:

$$\hat{\lambda}_\beta^{-(s)} = \frac{-\sum_{n=1}^N \sum_{t=1}^T \sum_{m=1}^M \mathbb{1}\{y_t \leq Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_1)^{(s)})\}}{\sum_{n=1}^N \sum_{t=1}^T \sum_{m=1}^M (y_t - Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_1)^{(s)}) \mathbb{1}\{y_t \leq Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_1)^{(s)})\}},$$

and

$$\hat{\lambda}_\beta^{+(s)} = \frac{\sum_{n=1}^N \sum_{t=1}^T \sum_{m=1}^M \mathbb{1}\{y_t > Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_Q)^{(s)})\}}{\sum_{n=1}^N \sum_{t=1}^T \sum_{m=1}^M (y_t - Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_Q)^{(s)}) \mathbb{1}\{y_t > Q_t^y(\cdot, \omega_{it}^{(m)}; \hat{\beta}(\tau_Q)^{(s)})\}}.$$

In this setting, it is computationally efficient to take $M = 1$ in the MCEM algorithm and report estimates of the average $\tilde{S} = S/2$ draws. This is known as the stochastic EM algorithm (stEM) of Celeux and Diebolt (1985). The sequence of maximizers $\hat{\theta}^{(s)}$ is a time-homogeneous Markov chain, which if ergodic, will converge to its stationary distribution. Nielsen (2000) provides sufficient conditions for ergodicity and provides asymptotic properties of the estimator when the "M-step" is solved using maximum likelihood. Arellano and Bonhomme (2016) discusses the asymptotic properties of the estimator when the M-step is solved using quantile regression in a panel dating setting, where for example $\omega_{it} = \omega_i$ would be modelled as a firm fixed-effect. They show that under correct specification of the parametric model, the estimator is root-N consistent and asymptotically normal. That is, assuming Q (the grid of the interpolating spline) and $J = J_y + J_\omega + J_{\omega_1} + J_\ell + J_m + J_\iota$ are fixed. Ideally, one would have these parameters grow with the sample size so that approximation error of the corresponding models approaches zero. Deriving the optimal rate of convergence in a nonparametric setting is an avenue for future work. Arellano *et al.* (2017) quantify the uncertainty of their estimates using parametric and nonparametric bootstrap clustered at the individual level. Both algorithms are computationally demanding, as it requires re-estimating the model many times. One alternative is to calculate confidence bands from the chain of parameters from the converged EM algorithm, which is common in the MCMC literature. These confidence bands are then interpreted as posterior intervals and are reported in Appendix C.

6 Application

The estimator is applied to data on U.S. manufacturing firms from the Standard and Poors Compustat database. The sample covers publicly traded firms and contains data from their financial statements. I collect a sample between 1997 and 2016 on sales, capital expenditures, property, plant, and equipment, employees, and other expenses to construct output, investment, capital stock, labor, and material inputs. The financial data is deflated using 3-digit deflators from the NBER-CES Manufacturing Industry Database. After data cleaning, there are a total of $N = 2961$ firms with an average of 1545 firms per year. Summary statistics are provided in Appendix A.

The estimation algorithm is ran with 500 random-walk Metropolis-Hastings steps and 300 EM steps, taking $M = 1$. In the sampling algorithm, productivity is drawn from a normal distribution centered at the current draw of productivity with variance equal to 0.01. This achieves an average acceptance rate of 10%. The production function is specified as before and I take $J_\omega = J_{\omega_1} = 3$ for the productivity processes, $J_l = J_\iota = 16$ for labor and investment demand, and $J_m = 27$ for materials demand. Therefore, the total number of parameters are $J = 85$. The final estimates are used to simulate productivity from its initial conditions and the decision rules for investment, labor, and materials. A capital accumulation process is needed to simulate these values. The process specified in Equation (11) is flexible and is used to accumulate capital following the perpetual inventory method with a constant depreciation rate set at 0.02.

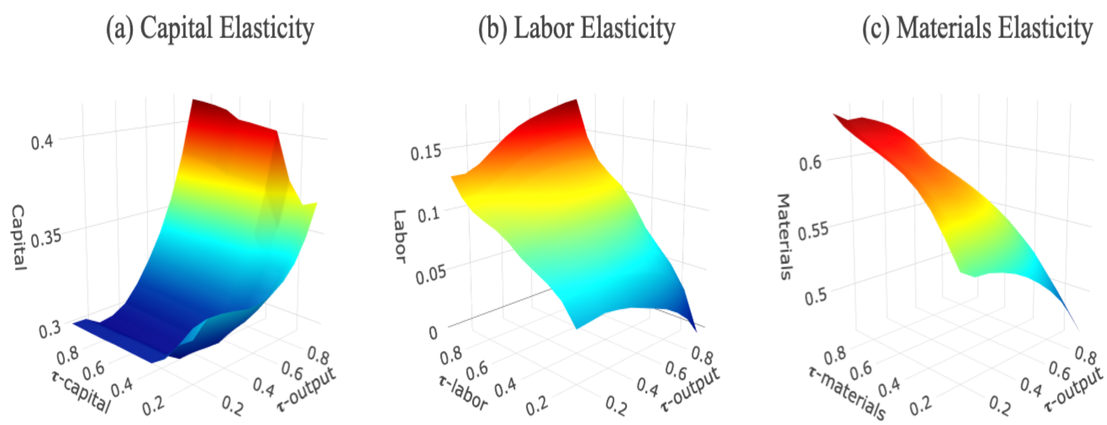
6.1 Empirical Results

6.1.1 Production Function Estimates

Estimates of the heterogeneous production function elasticities are shown in Figures 1 and 2. Panel (a) of Figure 1 reports the estimates of the average capital elasticity evaluated at percentiles of capital and output. The estimates range from 0.3 for firms at the lowest percentile of output and highest percentile of capital to 0.413 for firms at the highest percentile of output and capital. For firms with high levels of capital, there is more heterogeneity across the output distribution, contrasted to the low heterogeneity for firms at the lower percentiles of capital. Panel (b) reports the average labor elasticity evaluated at percentiles of labor and output. The relationship is opposite to that of capital. The estimates are -0.005 for firms at the highest percentile of output and lowest percentile of labor, and 0.179 for firms in the highest percentile of output and labor. For these estimates, there is larger heterogeneity across firms with low levels of labor than firms who use more labor. Panel (c) shows the estimates of the average materials elasticity evaluated at percentiles of materials and output. The relationship is similar to labor. The estimates are lowest at 0.467 for firms at the highest percentile of output and lowest percentile of materials, and highest at 0.634

for firms who use high levels of materials at the bottom of the output distribution. Overall, these results suggest that elasticities vary over the size of the firm measured by the rank on the conditional output distribution and the amount of inputs a firm uses.

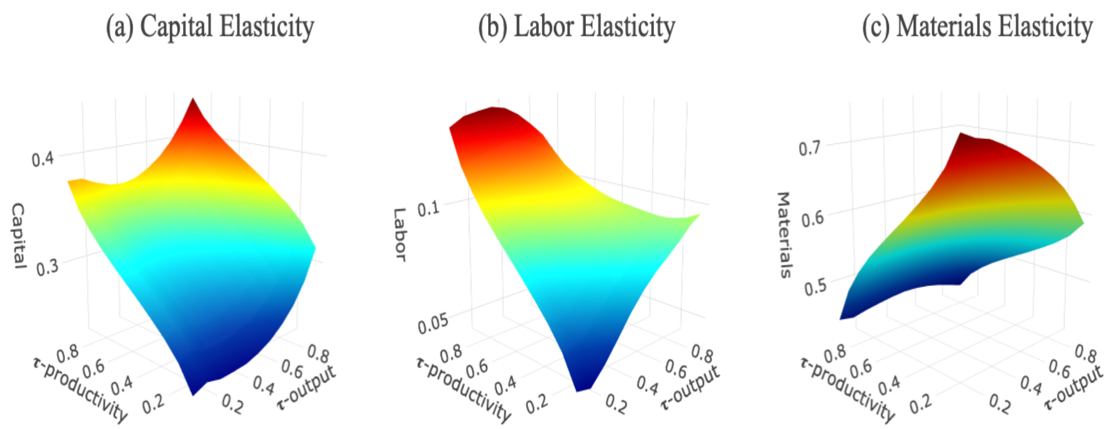
Figure 1: Output Elasticities



*Panel (a): Capital elasticity evaluated at τ_η and percentiles of capital τ_k averaged over values of ω_{it} and (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor elasticity evaluated at τ_η and percentiles of labor τ_l averaged over values of ω_{it} and (k_{it}, m_{it}). Panel (c): Materials elasticity evaluated at τ_η and percentiles of materials τ_m averaged over values of ω_{it} and (k_{it}, l_{it}).

In Figure 2, I report similar estimates as Figure 1, instead evaluating the estimates at fixed percentiles of productivity. Panel (a) reports the capital elasticities over percentiles of output and productivity. These results are more heterogeneous for firms with different levels of productivity. For low output and productivity firms, the capital elasticity is 0.233. For high output and productivity firms, this is 0.44. For low to medium productivity firms, these estimates are increasing faster in the rank of the output distribution, but for high productivity firms this relationship is U-shaped. For low output firms, capital elasticity increases faster when the firm is more productive. For high output firms, the rate at which the estimates increase is slower. These results imply that unobserved productivity is an important dimension of firm heterogeneity in capital elasticities. Panel (b) reports the estimates of labor elasticity. For low output and productivity firms, the estimate is 0.046 and for low output and high productivity firms, the estimate is 0.132. Unlike panel (b) in Figure 1, estimates are increasing in output size for low productivity firms and decreasing for firms at the highest percentile of productivity. Labor elasticity estimates are increasing in productivity except for high output firms, where the estimates are constant. Estimates rise faster for low output firms than high output firms. Panel (c) reports the material elasticities over percentiles of productivity. For low output and productivity firms, estimates are highest at 0.747, and for high output and productivity firms, estimates are lowest at 0.424. For high productivity firms, material elasticities are constant in output size, but decreasing for low productivity firms. Overall, these estimates are increasing in productivity size for fixed levels of output, although estimates increase faster for low output firms compared to higher output firms.

Figure 2: Output Elasticities (Over Productivity)

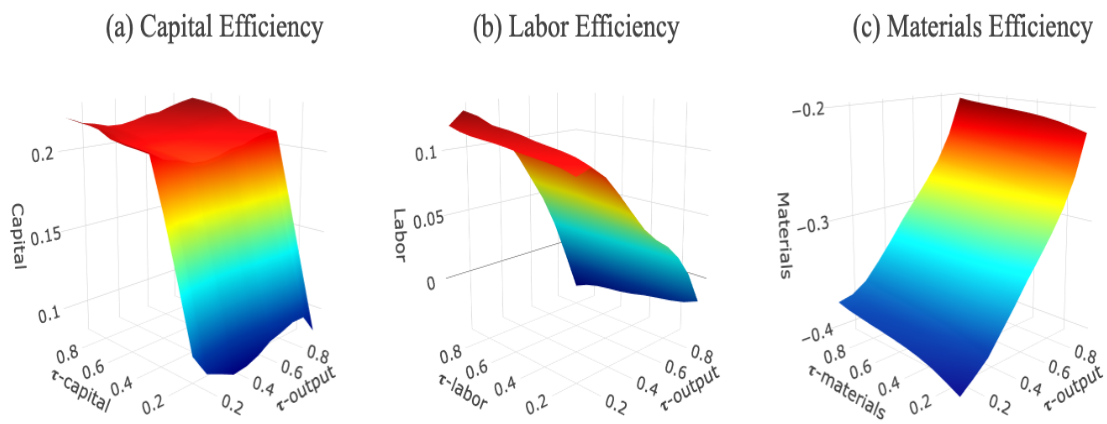


*Panel (a): Capital elasticity evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity. Panel (b): Labor elasticity evaluated at evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity. Panel (c): Materials elasticity evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity.

Figure 3 presents estimates of the non-Hicks neutral effects of productivity on capital, labor, and material inputs. These effects can be interpreted as the marginal productivity of inputs that follows from a small change in productivity. The labor-augmenting aspect of the shock is of particular importance, since the empirical literature often points to labor productivity as sources of long-run economic growth. Despite its importance, there are relatively few papers that study these sources of productivity at the firm-level. This is because identification and estimation of these models are difficult due to the issues of endogeneity and multi-dimensional productivity.[3] It is worth noting that the identification arguments presented here may accommodate multi-dimensional unobservables, such as Hicks-neutral and labor-augmenting productivity. Extra unobservables require additional proxies, which increases the data requirements in my approach, however the estimates under this alternative would be more suited for comparison to existing empirical work.

[3]As mentioned earlier, Doraszelski and Jaumandreu (2018) have made progress in this regard using rich firm-level data. Dermirer (2020) also studies nonparametric identification of these models and applies his estimates to U.S. public manufacturing firms.

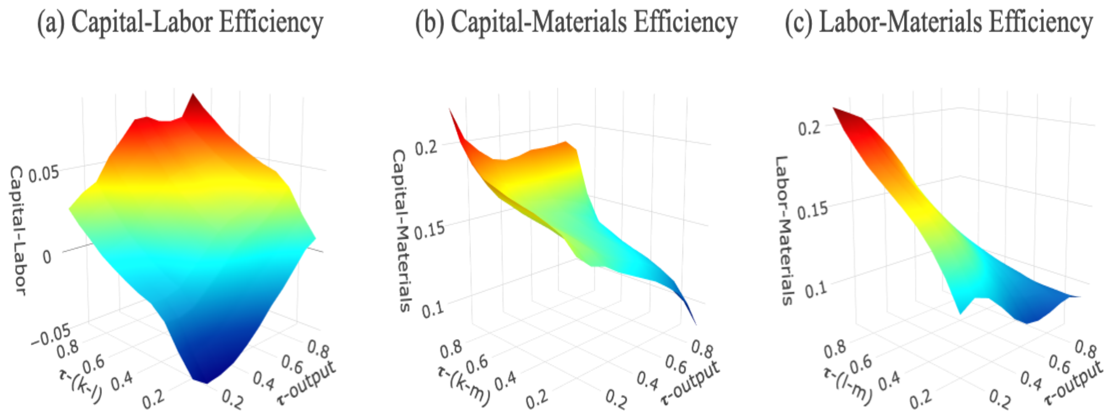
Figure 3: Effect of Productivity on Output Elasticities



*Panel (a): Capital efficiency evaluated at τ_η and percentiles of capital τ_k averaged over values of (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor efficiency evaluated at τ_η and percentiles of labor τ_l averaged over values of (k_{it}, m_{it}). Panel (c): Materials efficiency evaluated at τ_η and percentiles of materials τ_m averaged over values of (k_{it}, l_{it}).

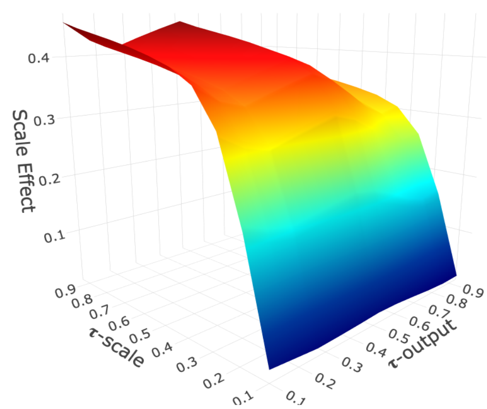
Panel (a) in Figure 3 shows the results for the capital-augmenting effect of productivity. These estimates are computed at various percentiles of output and capital to examine how the capital efficiency effect varies over firms. The estimates range from 0.086 for firms at the highest percentiles of output and lowest percentile of capital, to 0.224 for firms at the highest percentiles of output and capital. Overall, the capital efficiency effects are increasing for firms who use more capital, but almost constant across the conditional output distribution. For the labor efficiency estimates in the panel (b), there is more heterogeneity between firms of different sizes who use varying amounts of labor. The estimates range from -0.044 for firms at the highest percentile of output and labor, to 0.119 for firms at the lower percentiles of output, but the highest percentile of labor. Interestingly, the labor estimates are decreasing in labor size for the smallest percentile of output but increasing at the largest percentile. These results seem consistent with empirical results that find large amounts of firm variation in labor-productivity, and suggests that smaller firms use labor more efficiently than larger firms in this sample. Panel (c) reports the material efficiency estimates. These range from -0.403 for firms at the lowest percentiles of output and materials, to -0.204 for firms at the highest percentiles of output and materials. These estimates reveal that for firms in this sample, an increase in productivity leads to a decrease in the marginal product of materials. This could suggest that either firms are inefficient in their usage of materials or that there are structural differences in materials productivity that are not captured by these marginal effects.

Figure 4: Effects of Input Composition on Productivity



*Panel (a): Capital-labor effect evaluated at τ_η and percentiles of capital-labor τ-(k-l). Panel (b): Capital-materials effect evaluated at τ_η and percentiles of capital-materials τ-(k-m). Panel (c): Labor-materials effect evaluated at τ_η and percentiles of labor-materials τ-(l-m).

Figure 5: Effect of Scale on Productivity



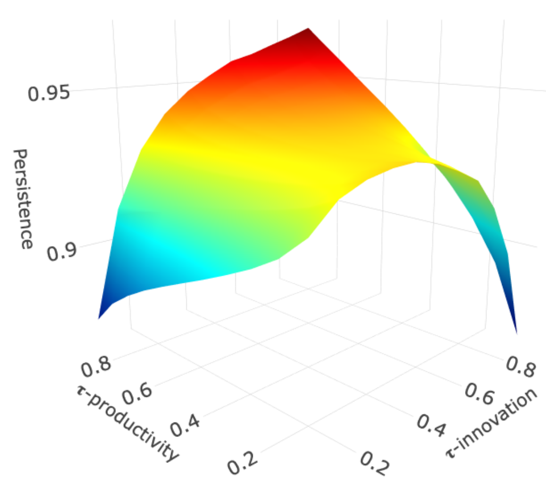
* Scale effect evaluated at τ_η and percentiles of scale (τ-scale).

Using Equation (16), I examine the role of input composition and scale of a firm on productivity. Similar to Navarro and Rivers (2018), I interpret these estimates as the effect of an increase in technological change through composition and scale on productivity. In Figure 4 panel (a), I find that small firms (as measured by low τ-output) and labor intensive firms (low τ-(k-l)) are less efficient at transforming increases in technological change into productivity compared to larger, more capital intensive firms. In fact, the estimates for these smaller, labor intensive firms are negative, which may suggest that the firms are inefficient in the use of capital relative to labor, or that technological change is labor-using as opposed to labor-saving. In panels (b) and (c), I find that large firms with high compositions of capital to materials and labor to materials are better at translating technological change into productivity gains. Finally, in Figure 5, I find that regardless of firm size, higher scale firms are more productive. This could be the case if high scale firms increase scale through higher capital usage. Since larger firms tend to be more capital intensive, the scale effect is positive and increasing.

6.1.2 Persistence of Productivity

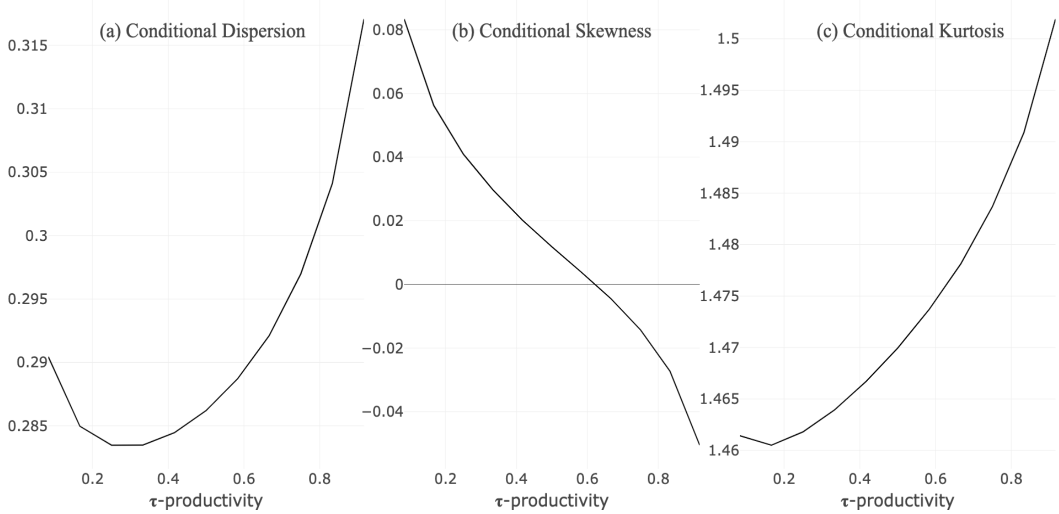
Next, I examine the estimates of the nonseparable productivity process. Figure 6 reports the estimates of productivity persistence at various percentiles of the innovation shock and percentiles of last period productivity. Persistence exhibits a significant asymmetric relationship. These results suggest that high productivity firms (τ-productivity$= 0.9$) hit by negative shocks have a lower persistence of productivity history (0.875) than low productivity firms (τ-productivity$= 0.1$) hit by the same negative shock (0.925). This indicates that for high productivity firms, large unanticipated negative shocks can reduce the history of high productivity by more than firms with a history of low productivity. This relationship changes when firms are hit by large positive shocks. High productivity firms have higher persistency of productivity history (0.968) than low productivity firms (0.870) when hit by positive shocks.

Figure 6: Productivity Persistence



*Estimates of average productivity persistence evaluated at τ_ξ and percentiles of previous productivity.

Figure 7: Higher Moments of the Conditional Productivity Distribution



* Panel (a): Conditional dispersion (second moment) evaluated at different percentiles of previous productivity. Panel (b): Conditional skewness (third moment) evaluated at different percentiles of previous productivity. Panel (c): Conditional kurtosis (fourth moment) evaluated at different percentiles of previous productivity

Using the estimates of the productivity process, I construct estimates of the higher-moments of the conditional productivity distribution in Figure 7. For a measure of uncertainty, I use a conditional quantile-based estimate of $\mu_{2,t}(\omega_{it-1}, \tau) = Q_t^\omega(\omega_{it-1}, \tau) - Q_t^\omega(\omega_{it-1}, 1-\tau)$ for some $\tau \in (1/2, 1)$. I evaluate uncertainty at different levels of ω_{it-1} fixed at $\tau = 10/11$ and find that productivity uncertainty is lowest for firms with a modest history of productivity. Uncertainty for low productivity firms is high, however higher productivity histories are associated with higher uncertainty. This evidence is complementary to that of Bloom *et al.* (2018) and shows that uncertainty varies firm-to-firm in addition to over time. In panel (b), I report conditional quantile-based estimates of productivity skewness given by the following equation:

$$\mu_{3,t}(\omega_{it-1}, \tau) = \frac{Q_t^\omega(\omega_{it-1}, \tau) + Q_t^\omega(\omega_{it-1}, 1-\tau) - 2Q_t^\omega(\omega_{it-1}, 1/2)}{\mu_2(\omega_{it-1}, \tau)}, \quad \tau \in (1/2, 1). \tag{33}$$

The estimate of conditional skewness captures whether the dispersion in productivity is driven by the left or right tail of the productivity distribution for different productivity histories. I find that skewness is positive for firms with low histories of productivity and negative for high histories. This suggests that firms with low histories have a higher probability of drawing values of current productivity to the right of the distribution after a positive productivity shock. Firms with high histories have a higher probability of drawing values to the left of the distribution, following a negative productivity shock. Overall, this result strengthens the argument that business cycles can generate asymmetries in the productivity distribution conditional on a firm's productivity history. In panel (c), I measure conditional kurtosis using

$$\mu_{4,t}(\omega_{it-1}, \tau, \gamma) = \frac{Q_t^{\omega}(\omega_{it-1}, \gamma) - Q_t^{\omega}(\omega_{it-1}, 1-\gamma)}{\mu_2(\omega_{it-1}, \tau)}, \quad \tau < \gamma, \tag{34}$$

with $\tau = 9/11$ and $\gamma = 10/11$. I find that kurtosis increases with productivity history. The results presented in this section illustrate the importance of firm-specific uncertainty and asymmetries in the productivity process. In the next section, these results will be used to show how firm responses to labor, investment, and input misallocation vary with productivity history.

6.1.3 Input Demand and Productivity

In this section, I examine how firms adjust inputs with respect to changes in productivity and innovation shocks. The input responses to productivity are calculated as the derivative of Equations (23), (24), and (25) with respect to productivity. These provide insights on how firms input demand changes in response to technological or organizational innovations measured by the unobserved productivity component. For example, whether a firm adjusts its labor demand in response to innovations in automation, has important consequences for employment displacement and its public policy responses. My estimates show that there is heterogeneity at the firm-level in these productivity responses at different percentiles of productivity and input demand.

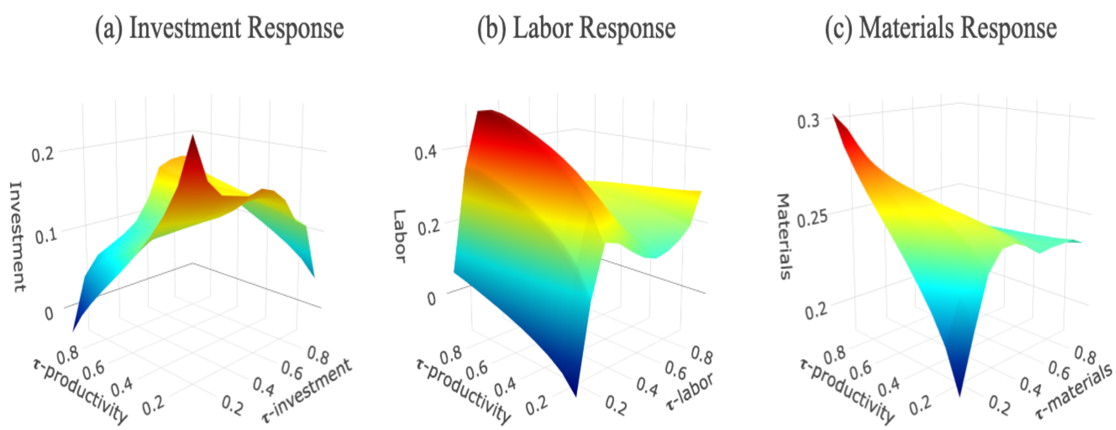
Panel (a) in Figure 8 shows the relationship between investment demand and productivity. Firms at the lowest percentile of investment and productivity have the largest productivity response at 0.247. As productivity increases for lower investment firms, this effect decreases to -0.033. For high investment firms at the lowest percentile of productivity, the effect is 0.039. For similar levels of investment, high productivity firms have an effect equal to 0.164. Overall, these results suggest there is significant heterogeneity in firms' investment adjustments with respect to changes in productivity levels.

Panel (b) shows the relationship between labor demand and productivity. Firms at the lowest percentile of labor and productivity have an effect equal to -0.114. For firms at the lowest percentile of labor and highest percentiles of productivity, the effect is equal to

0.06. For firms at the highest percentile of labor, but the lowest percentile of productivity, the effect approaches 0.288, but for firms at highest percentile of productivity this effect decreases to 0.236. The shape of the labor productivity response is an inverse U-shape for low labor firms. For firms who use more labor, the estimates are somewhat flatter across the productivity distribution. These results show that for firms who use less labor and are less productive, increases in productivity leads to an increase in the amount of labor, whereas large labor firms do not adjust labor as much in response to productivity changes at different levels of productivity.

Panel (c) shows the relationship between material input demand and productivity. Firms at the lowest percentile of materials and productivity have a productivity effect equal to 0.185. For firms at the lowest percentile of materials and highest percentiles of productivity, this effect is equal to 0.303. For firms at the highest percentile of materials, but the lowest percentile of productivity, the effect is equal to 0.235, and for the highest percentile of productivity is 0.226. For firms who use the smallest amounts of materials and who are not productive, the productivity effect is smallest. The effect is largest for higher productivity firms. Overall, firms respond to productivity increases by using more material inputs.

Figure 8: Input Demand Responses to Productivity



*Panel (a): Investment demand evaluated at τ_ζ and percentiles of productivity τ_ω averaged over values of k_{it}. Panel (b): Labor demand evaluated at τ_{ϵ_l} and percentiles of productivity τ_ω averaged over values of k_{it}. Panel (c): Material demand evaluated at τ_{ϵ_m} and percentiles of productivity τ_ω averaged over values of k_{it} and l_{it}.

6.1.4 Impulse Responses to Productivity Shocks

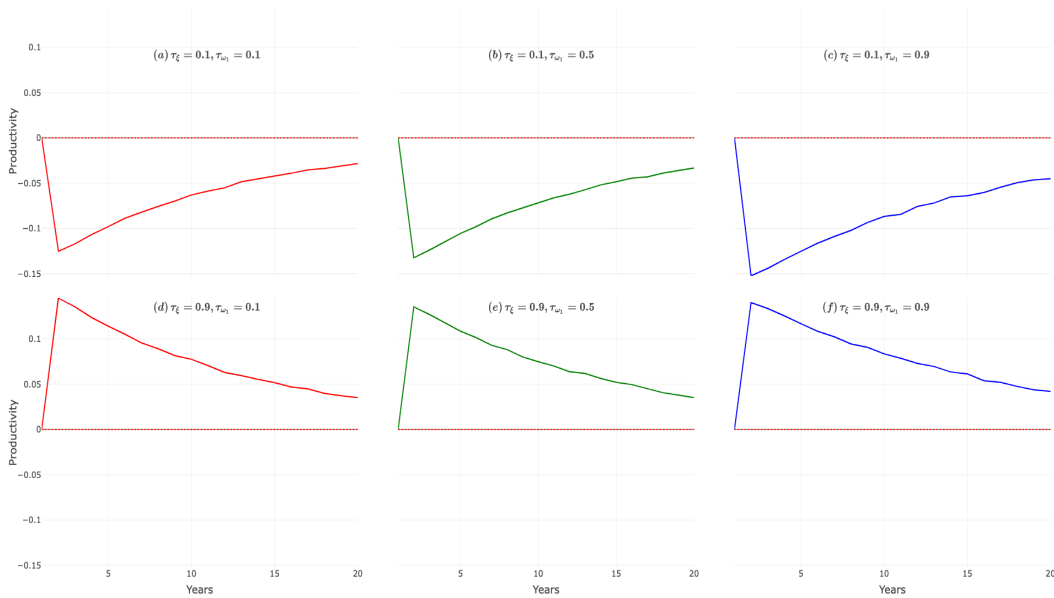
This section simulates the impact of innovation shocks to the productivity process and input demand functions using estimates from the model. Similar to HHS, I estimate how quickly firms respond to shocks to current productivity. This analysis will show whether input decision rules for capital, labor, and materials are subject to substantial adjustment frictions. For example, if the finding is that labor responds positively to increases in productivity, then policies designed to increase productivity may have a faster effect depending on how quickly the firm is able to adjust its work force, which has implications for labor market outcomes. My model allows me to examine this effect on two different dimensions: the size of the labor demand across firms and the size of the productivity shock. This estimator can be given by:

$$\hat{l}(\tau_\xi, \tau_{\epsilon_l}) = \hat{\mathbb{E}}\left[\frac{\partial Q_t^\ell\big(k_{it}, Q_t^\omega(\omega_{it-1}, \tau_\xi), \tau_{\epsilon_l}\big)}{\partial \omega_{it}} \times \left(\frac{\partial Q_t^\omega(\omega_{it-1}, \tau_\xi)}{\partial \xi_{it}}\right)\right],$$

where $\partial Q_t^\omega(\omega_{it-1}, \tau_\xi)/\partial \xi_{it}$ can be approximated by finite differences. In practice, I simulate impulse response functions under various innovation shocks to productivity and input demand functions under some initial conditions.

Figures 9, 10, 11 and 12 report median differences in low innovation shocks $\tau_\xi = 0.1$ and high innovation shocks $\tau_\xi = 0.9$ and firms hit by medium innovation shocks at $\tau_\xi = 0.5$ for productivity, capital, labor and materials. I simulate the model so that the impact of the shock occurs at $t = 2$. I examine the initial responses to productivity and inputs, as well as the length of time it takes for firms to recover from negative productivity shocks. This analysis is somewhat similar to HHS. In their paper, they study how quickly firms adjust inputs in response to the latest shocks in productivity. Their GMM estimator allows them to estimate the covariance between inputs, productivity, and its shocks. This is useful in their context, as it provides guidance for choosing proxies for the latent productivity. These estimates can also identify industry efficiency and frictions in the input markets. Unlike their GMM estimator, my estimates document the impact of differently sized innovation shocks and input demand functions beyond the mean, as well as the full history of the impact. This analysis is also similar to Arellano *et al.* (2017), who study the impact of earnings shocks on the process for income and consumption. My application further illustrates the importance of an asymmetric productivity process and its consequences for the projection of future productivity growth, input demand, and misallocation.

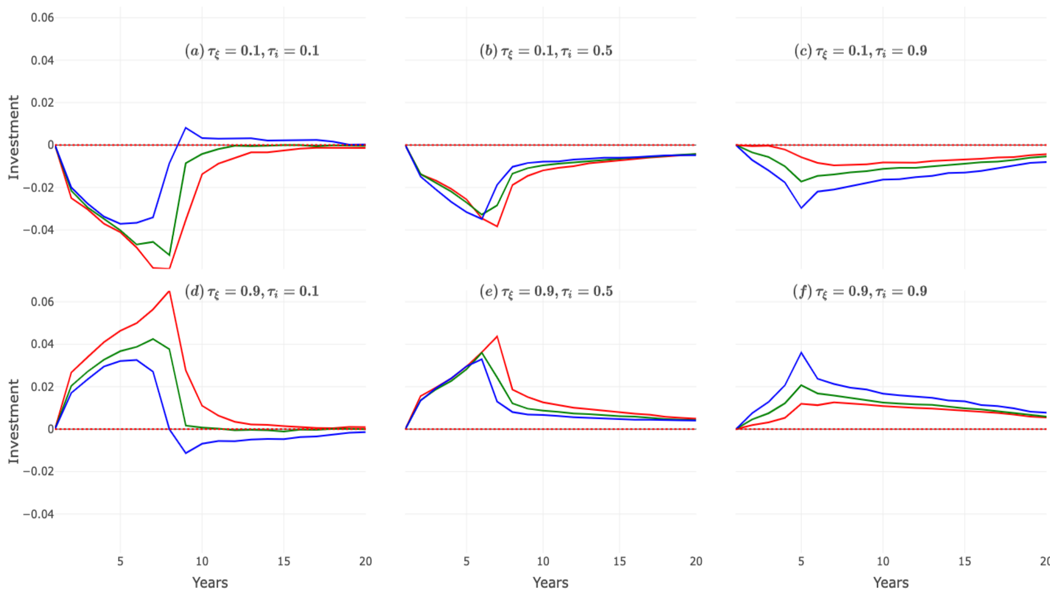
Figure 9: Impulse Response of an Innovation Shock to Productivity



*Top row: Differences in productivity between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in productivity between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity.

The productivity responses to innovation shocks are reported in Figure 9, which shows the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and large positive shock ($\tau_\xi = 0.9$) in panel (d-f) for various levels of initial productivity $\tau_{\omega_1} = (0.1, 0.5, 0.9)$. For firms with the lowest initial productivity (red), a large negative innovation shock decreases productivity by 12.5%, while a large positive shock increases productivity by 14.5%. For firms with median initial productivity (green), a large negative innovation shock decreases productivity by 13.2%, and a large positive shock increases productivity by about 13.5% For firms with the highest initial productivity (blue), a large negative innovation shock decreases productivity by 15.2%, and a large positive shock increases productivity by about 14%. There is no observable difference in the length of time required to recover from negative productivity shocks, which is consistent with the small difference in productivity persistence for high and low productivity firms hit by negative innovation shocks. Another conclusion is that firms with low productivity history have an asymmetric gain in productivity following a positive shocks compared to firms with high productivity history who have an asymmetric loss.

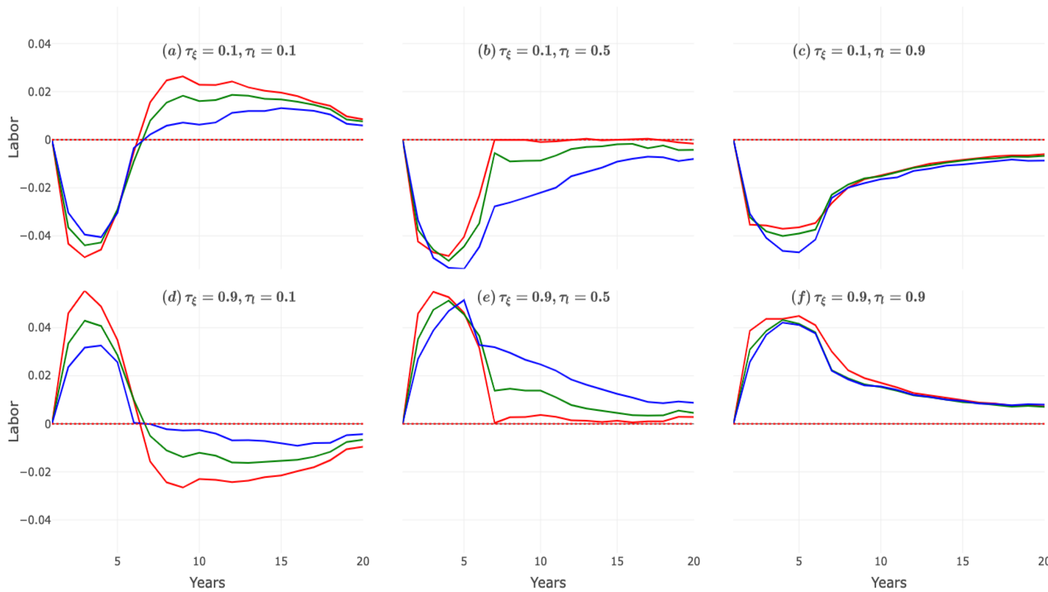
Figure 10: Impulse Response of an Innovation Shock to Investment



*Top row: Differences in investment between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand. Bottom row: Differences in investment between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

The investment responses to innovation shocks are reported in Figure 10, which shows the impact of negative productivity shocks in panel (a-c) and positive productivity shocks in panel (d-f) for various levels of investment demand $\tau_i = (0.1, 0.5, 0.9)$ and initial productivity. The red, green, and blue lines correspond to firms with initial productivity levels $\tau_{\omega_1} = (0.1, 0.5, 0.9)$. For firms with the lowest investment demand and initial productivity, a large negative productivity shock initially decreases investment by 2.5% and decreases until year eight to 5.8%. For high investment and initial productivity, the initial drop is much lower at 0.7% and the largest drop occurs earlier at year 5 at 3%. Overall, low investment firms adjust capital investment more dramatically in response to a negative productivity shock and the largest decreases occur for low productivity firms. High investment firms responses are smoother and less dramatic, and unlike low investment firms, firms with higher initial productivity face a larger decrease in investment. Overall, this could suggest the presence of high adjustment costs that penalize large investment changes at $\tau_i = 0.9$ relative to small changes at $\tau_i = 0.1$.

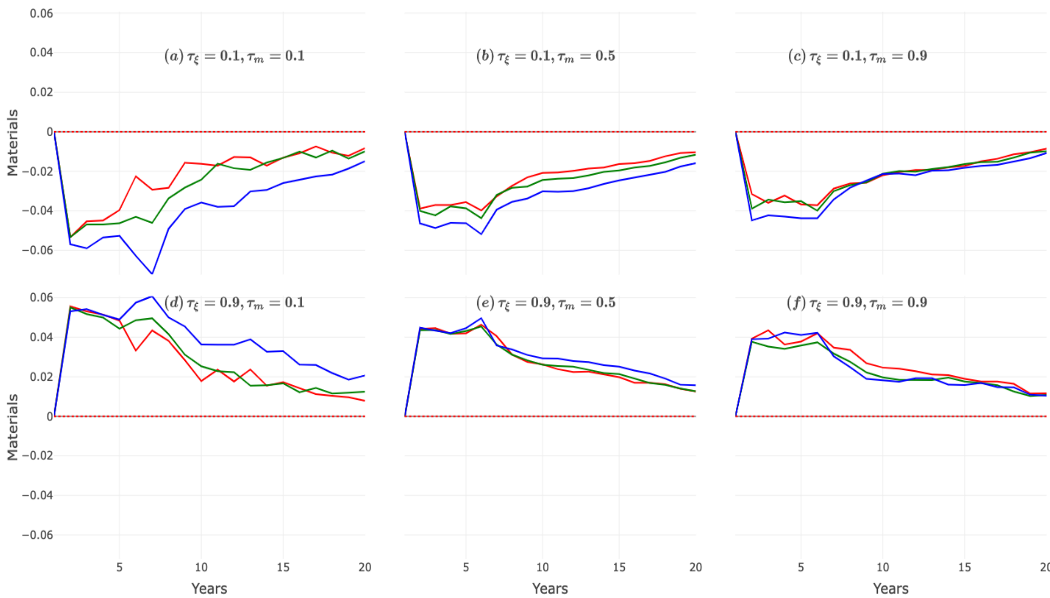
Figure 11: Impulse Response of an Innovation Shock to Labor



*Top row: Differences in labor between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand. Bottom row: Differences in labor between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

The labor responses to innovation shocks are reported in Figure 11, which shows the impact of a negative productivity shock in panel (a-c) and a positive productivity shock in panel (d-f) for various levels of labor demand $\tau_\ell = (0.1, 0.5, 0.9)$ and initial productivity. For low labor firms in panel (a) and (d) there is an overshoot and undershoot of labor demand in the medium term following a negative and positive productivity shock. These findings are similar to that of Bloom (2009) who finds that medium term hiring occurs after an increase in the volatility of business conditions. Therefore, one possible explanation for these results is an increase in productivity volatility following a negative shock that causes some firms near the hiring threshold to hire in response to a positive shock and other firms to do nothing in response to a negative shock. The opposite situation may arise in panel (d) if the positive shock leads to a decrease in productivity volatility. This overshooting/undershooting phenomenon does not occur for firms at the higher end of the labor demand distribution.

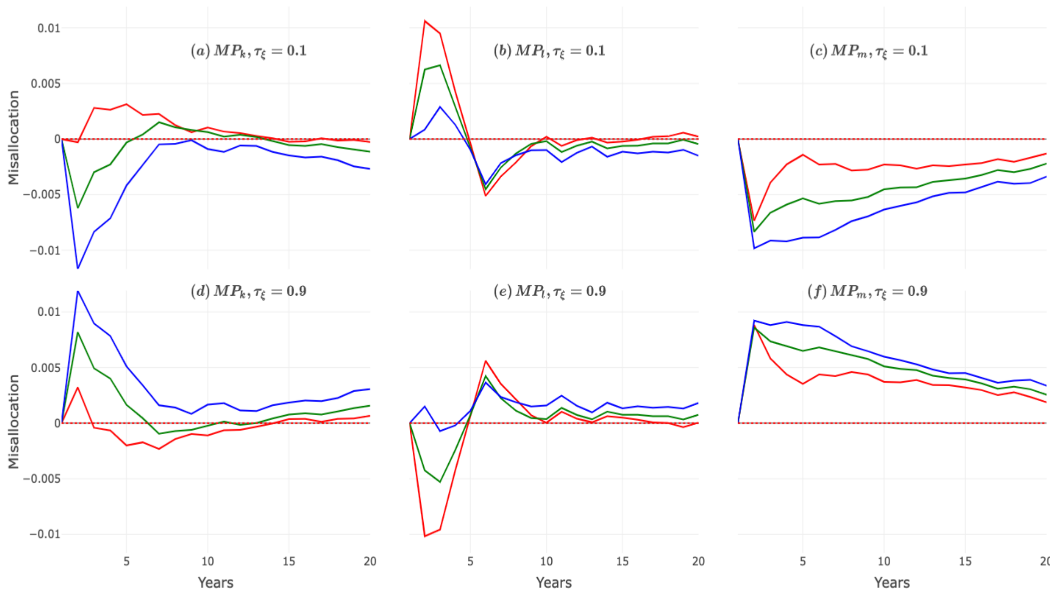
Figure 12: Impulse Response of an Innovation Shock to Materials



*Top row: Differences in materials between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand. Bottom row: Differences in materials between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

The materials responses to innovation shocks are reported in Figure 12, which shows the impact of a negative productivity shock in panel (a-c) and a positive productivity shock in panel (d-f) for various levels of materials demand $\tau_m = (0.1, 0.5, 0.9)$ for different levels of initial productivity. For firms with the lowest materials demand and highest initial productivity, a large negative productivity shock decreases material inputs by 5.7%, while a large positive shock increases material inputs by 5.5%. For firms with the highest materials demand, there is not much heterogeneity between different levels of initial productivity. Material demand falls around 4.5% in response to a negative shock and rises 3.9% in response to a positive shock.

Figure 13: Impulse Response of an Innovation Shock to Input Misallocation



*Top row: Differences in marginal product dispersion between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in marginal product dispersion between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

In Figure 13, I examine the response of input misallocation to productivity shocks. Misallocation is measured as the cross-sectional standard deviation in marginal products of inputs. Several papers have sought to identify the sources of misallocation. David and Venkateswaran (2019) develop a unifying framework to distinguish sources of capital misallocation, such as uncertainty, adjustment costs, and financial frictions. Uncertainty plays a role in labor misallocation as well. Bloom *et al.* (2018) shows that an uncertainty shock leads to a 15% increase in the dispersion of the marginal product of labor. Misallocation of intermediate inputs, such as materials, has not received much attention, although Boehm and Oberfield (2020) finds that mechanisms such as inefficient contract enforcement play a key role in input choices and misallocation in India. There does not seem to be any existing evidence suggesting uncertainty causes misallocation in intermediate inputs. In the left column of Figure 13, I plot the path of capital misallocation following a negative and positive productivity shock. One interesting phenomenon is that misallocation falls for medium and high productivity firms following a negative shock, but rises for low productivity firms. A rise in misallocation follows a large positive productivity shock for all levels of productivity. In the middle column, there is an increase in labor misallocation following a negative shock and a decrease following a positive shock. The right column plots material misallocation and shows that a negative shock leads to a decrease in misallocation and a positive shock leads to an increase in misallocation for all levels of productivity.

7 Conclusion

This paper proposes a nonseparable model for firm production, which allows for elasticities and non-Hicks neutral effects of productivity to vary over the conditional distribution of output. The estimates reveal substantial heterogeneity across this distribution, as well as across different percentiles of input demand. This challenges the standard approach of estimating production functions, which specify technology that is fixed across firms, and instead suggests that nonseparable, firm-specific models are more suitable when heterogeneity is prevalent in the data. The approach considered here also allows for a more flexible productivity process, where persistence in productivity history can vary with respect to the latest innovation shocks, and that good or bad shocks have asymmetric impacts for both high and low productivity firms.

The production function, input demand functions, and productivity are nonparametrically identified in the presence of nonseparable unobservables. I show that under additional independence restrictions, conditional quantile restrictions can be imposed, and the quantile estimators can be used to capture firm-level heterogeneity. The estimator proposed in this paper is computationally tractable and involves quantile regression in each iteration of the simulation algorithm. This provides new results that have not been considered in the prior production function literature. For example, this paper shows that firms have asymmetric input adjustments in response to productivity changes. This type of analysis is useful from a policy perspective, as proposals aimed to increase productivity may have different outcomes for firms with different input demand functions and productivity levels. This paper also studies the adjustment frictions of input demand functions in response to innovation shocks to productivity and finds asymmetries in the impacts of good and bad shocks. The overall finding is that firms with the highest input-productivity adjustments also have the largest drop in input demand following a bad productivity shock. For example, I found that low investment firms with low productivity have the largest decrease in investment following a negative productivity shock and adjust investment more rapidly than high investment firms.

There are many interesting extensions that can be considered in the framework proposed in this paper. The first would be to include additional unobservables beyond the productivity term. For example, fixed effects can be included in the production function and productivity process to account for firm-specific unobservables. The current model assumes productivity is scalar and that its interactions with inputs measure the magnitude of non-Hicks neutral effects. It would be interesting to consider multi-dimensional productivity shocks, for example a Hicks-neutral and a labor-augmenting term to capture productivity effects that are biased towards labor. Extending the identification arguments to this case would be more demanding since labor-augmenting productivity is typically serially correlated. Lastly, the results presented here are often used to estimate other aspects of firm technology and market power. Further analysis of total factor productivity and markup estimates would provide an interesting comparison with results from the standard production function model.

References

Ackerberg, D., Benkard, C. L., Berry, S. and Pakes, A. (2007). Chapter 63 econometric tools for analyzing market outcomes. In *Handbook of Econometrics*, Elsevier, pp. 4171–4276.

— and Hahn, J. (2015). Some non-parametric identification results using timing and information set assumptions. Working paper.

Ackerberg, D. A., Caves, K. and Frazer, G. (2015). Identification properties of recent production function estimators. *Econometrica*, **83** (6), 2411–2451.

Arellano, M., Blundell, R. and Bonhomme, S. (2017). Earnings and consumption dynamics: A nonlinear panel data framework. *Econometrica*, **85** (3), 693–734.

— and Bonhomme, S. (2016). Nonlinear panel data estimation via quantile regressions. *The Econometrics Journal*, **19** (3), C61–C94.

— and — (2017). Quantile selection models with an application to understanding changes in wage inequality. *Econometrica*, **85** (1), 1–28.

Balat, J., Brambilla, I. and Sasaki, Y. (2018). Heterogeneous firms: skilled-labor productivity and the destination of exports, Working paper.

Bloom, N. (2009). The impact of uncertainty shocks. *Econometrica*, **77** (3), 623–685.

—, Floetotto, M., Jaimovich, N., Saporta-Eksten, I. and Terry, S. J. (2018). Really uncertain business cycles. *Econometrica*, **86** (3), 1031–1065.

Blundell, R., Horowitz, J. and Parey, M. (2017). Nonparametric estimation of a nonseparable demand function under the slutsky inequality restriction. *The Review of Economics and Statistics*, **99** (2), 291–304.

Boehm, J. and Oberfield, E. (2020). Misallocation in the market for inputs: Enforcement and the organization of production. *The Quarterly Journal of Economics*, **135** (4), 2007–2058.

Celeux, G. and Diebolt, J. (1985). The sem algorithm: a probabilistic teacher algorithm derived from the em algorithm for the mixture problem. *Computational Satistics Quarterly*, (2), 73–82.

Chernozhukov, V., Fernndez-Val, I. and Galichon, A. (2010). Quantile and probability curves without crossing. *Econometrica*, **78** (3), 1093–1125.

David, J. M. and Venkateswaran, V. (2019). The sources of capital misallocation. *American Economic Review*, **109** (7), 2531–2567.

Dermirer, M. (2020). Production function estimation with factor augmenting technology: An application to markups. Working paper.

Doraszelski, U. and Jaumandreu, J. (2013). R&d and productivity: Estimating endogenous productivity. *The Review of Economic Studies*, **80** (4), 1338–1383.

— and — (2018). Measuring the bias of technological change. *Journal of Political Economy*, **126** (3), 1027–1084.

Dunford, N. and Schwartz, J. T. (1971). *Linear Operators*, vol. 3. Wiley.

Gandhi, A., Navarro, S. and Rivers, D. A. (2020). On the identification of gross output production functions. *Journal of Political Economy*, **128** (8), 2973–3016.

Hu, Y., Huang, G. and Sasaki, Y. (2020). Estimating production functions with robustness against errors in the proxy variables. *Journal of Econometrics*, **215** (2), 375–398.

— and Schennach, S. M. (2008). Instrumental variable treatment of nonclassical measurement error models. *Econometrica*, **76** (1), 195–216.

Kasahara, H., Schrimpf, P. and Suzuki, M. (2017). Identification and estimation of production function with unobserved heterogeneity. Working paper.

Kim, K. I., Petrin, A. and Song, S. (2016). Estimating production functions with control functions when capital is measured with error. *Journal of Econometrics*, **190** (2), 267–279.

Levinsohn, J. and Petrin, A. (2003). Estimating production functions using inputs to control for unobservables. *Review of Economic Studies*, **70** (2), 317–341.

Li, T. and Sasaki, Y. (2017). Constructive identification of heterogeneous elasticities in the cobb-douglas production function. Working paper.

Navarro, S. and Rivers, D. A. (2018). Nonparametric identification of productivity in nonseparable production functions. Working paper.

Nielsen, S. F. (2000). The stochastic EM algorithm: Estimation and asymptotic results. *Bernoulli*, **6** (3), 457–489.

Olley, G. S. and Pakes, A. (1996). The dynamics of productivity in the telecommunications equipment industry. *Econometrica*, **64** (6), 1263–1297.

Portnoy, S. and Koenker, R. (1997). The gaussian hare and the laplacian tortoise: computability of squared-error versus absolute-error estimators. *Statistical Science*, **12** (4), 279–300.

Salgado, S., Guvenen, F. and Bloom, N. (2019). Skewed business cycles. Working paper.

Wei, Y. and Carroll, R. J. (2009). Quantile regression with measurement error. *Journal of the American Statistical Association*, **104** (487), 1129–1143.

Appendix

A Data Appendix

Table 1: Summary Statistics (in logs) for U.S. Manufacturing Firms

	1st Qu.	Median	3rd Qu.	Mean	sd
Output	4.24	5.79	7.27	5.79	2.14
Capital	3.12	4.84	6.45	4.81	2.35
Labor	-1.23	0.22	1.62	0.21	1.95
Materials	3.95	5.47	6.95	5.46	2.15
Investment	0.57	2.40	3.94	2.23	2.49

Variable Construction:

- Output: Deflated Net Sales from Compustat (SALE).
- Capital: Deflated Property Plant and Equipment Net of Depreciation (PPENT).
- Labor: Number of Workers (EMPLOY).
- Labor Expense: EMPLOY times average industry wage calculated from the ratio of PAY and EMP in the NBER-CES Manufacturing Industry Database.
- Materials: Deflated Sales (SALE)-Operating Income Before Depreciation (OIBDP)-labor expense.
- R&D: XRD in Compustat.

B Identification

In this section, I show how the results of Hu and Schennach (2008) can be applied to identify the production function, input demand functions, and the marginal distribution of productivity. Technical details for the proof of their decomposition technique can be found in their paper.

Proof of Theorem 3.1 :
First, a conditional density constructed from observed data can be written as a product of the unknown conditional densities of interest:

$$\begin{aligned}
f_{y_t,I_t|y_{t+1},Z_t} &= \int f_{y_t,I_t,\omega_t|y_{t+1},Z_t}(y_t,I_t,\omega_t|y_{t+1},Z_t)d\omega_t \\
&= \int f_{y_t|y_{t+1},I_t,\omega_t,Z_t}(y_t|y_{t+1},I_t,\omega_t,Z_t)f_{I_t|y_{t+1},\omega_t,Z_t}(I_t|y_{t+1},\omega_t,Z_t)f(\omega_t|y_{t+1},Z_t)d\omega_t \\
&= \int f_{y_t|\omega_t,Z_t}(y_t|\omega_t,Z_t)f_{I_t|\omega_t,Z_t}(I_t|\omega_t,Z_t)f(\omega_t|y_{t+1},Z_t)d\omega_t,
\end{aligned} \tag{35}$$

where the third line follows from applying the conditional independence in Assumption 3.1. The goal of the identification strategy is to show that the conditional densities in Equation (35) can be written into its corresponding integral operators, which can be shown to admit a unique decomposition. Using Definition 3.1 and omitting the conditioning on Z_t for notational convenience:

$$\begin{aligned}
[L_{y_t,I_t|y_{t+1}}g](y_t) &= \int f_{y_t,I_t|y_{t+1}}(y_t,I_t|y_{t+1})g(y_{t+1})dy_{t+1} \\
&= \int\int f_{y_t,I_t,\omega_t|y_{t+1}}(y_t,I_t,\omega_t|y_{t+1})d\omega_t g(y_{t+1})dy_{t+1} \\
&= \int\int f_{y_t|I_t,y_{t+1},\omega_t}(y_t|I_t,y_{t+1},\omega_t)f_{I_t|y_{t+1},\omega_t}(I_t|y_{t+1},\omega_t)f_{\omega_t|y_{t+1}}(\omega_t|y_{t+1})g(y_{t+1})dy_{t+1}d\omega_t \\
&= \int f_{y_t|,\omega_t}(y_t|\omega_t)f_{I_t|\omega_t}(I_t|\omega_t)\int f_{\omega_t|y_{t+1}}(\omega_t|y_{t+1})g(y_{t+1})dy_{t+1}d\omega_t \\
&= \int f_{y_t|\omega_t}(y_t|\omega_t)f_{I_t|\omega_t}(I_t|\omega_t)[L_{\omega_t|y_{t+1}}g](\omega_t)d\omega_t \\
&= \int f_{y_t|\omega_t}(y_t|\omega_t)[\Delta_{I_t|\omega_t}L_{\omega_t|y_{t+1}}g](\omega_t)d\omega_t \\
&= [L_{y_t|\omega_t}\Delta_{I_t|\omega_t}L_{\omega_t|y_{t+1}}g](\omega_t),
\end{aligned}$$

where $\Delta_{I_t|\omega_t}$ is the diagonal operator mapping $g(\omega_t)$ to the function $f_{I_t|\omega_t}(I_t|\omega_t)g(\omega_t)$. Therefore, the following are equivalent:

$$L_{y_t,I_t|y_{t+1}} = L_{y_t|\omega_t}\Delta_{I_t|\omega_t}L_{\omega_t|y_{t+1}}. \tag{36}$$

Integrating (36) over I_t yields $L_{y_t|y_{t+1}} = L_{y_t|\omega_t}L_{\omega_t|y_{t+1}}$. Then using Assumption 3.2:

$$L_{\omega_t|y_{t+1}} = L^{-1}_{y_t|\omega_t}L_{y_t|y_{t+1}}. \tag{37}$$

Plugging (37) into (36):

$$L_{y_t,I_t|y_{t+1}} = L_{y_t|\omega_t}\Delta_{I_t|\omega_t}(L^{-1}_{y_t|\omega_t}L_{y_t|y_{t+1}}).$$

Note that the operator $L_{y_t|y_{t+1}} = L_{y_t|\omega_t}L_{\omega_t|y_{t+1}}$ is injective due to Assumption 3.2. Then we have the following:[4]

$$L_{y_t,I_t|y_{t+1}}L^{-1}_{y_t|y_{t+1}} = L_{y_t|\omega_t}\Delta_{I_t|\omega_t}L^{-1}_{y_t|\omega_t}. \tag{38}$$

The LHS of (38) is a function of observed data, which can be considered as known. This expression states that the LHS admits a spectral decomposition that takes the form of an eigenvalue-eigenfunction decomposition. To identify the unobserved densities of interest, the representation in (38) and its decomposition must be unique. This is guaranteed by Theorem XV.4.5 in Dunford and Schwartz (1971) and Assumptions 3.3 and 3.4. Then applying Theorem 1 in Hu and Schennach (2008) identifies $f_{y_t|\omega_t,Z_t}$, $f_{I_t|\omega_t,Z_t}$ and $f_{\omega_t|y_{t+1},Z_t}$.

The marginal distribution of productivity is identified from

$$f_{\omega_t} = \int f_{y_{t+1},\omega_t}dy_{t+1} = \int f_{\omega_t|y_{t+1}}f_{y_{t+1}}dy_{t+1},$$

since $f_{y_{t+1}}$ is observed and $f_{\omega_t|y_{t+1}}$ was identified from Theorem 3.1. The input demand functions for m_t and l_t are identified since f_{ω_t} is known. The next step is identification of the Markov process $f_{\omega_{t+1}|\omega_t}$ using Corollary 3.1 and 3.2.

Proof of Corollary 3.1:

Note that the integral operator corresponding to the density $f_{y_{t+1}|\omega_t}(y_{t+1}|\omega_t)$ can be written

[4] The fact that injectivity of $L_{y_{t+1}|\omega_t}$ implies injectivity of $L_{\omega_t|y_{t+1}}$ is non-trivial, but is guaranteed from Lemma 1 in Hu and Schennach (2008).

as:

$$
\begin{aligned}
[L_{y_{t+1}|\omega_t} g](y_{t+1}) &= \int f_{y_{t+1}|\omega_t}(y_{t+1}|\omega_t) g(\omega_t) d\omega_t \\
&= \int \int f_{y_{t+1},\omega_{t+1}|\omega_t}(y_{t+1}, \omega_{t+1}|\omega_t) d\omega_{t+1} g(\omega_t) d\omega_t \\
&= \int f_{y_{t+1}|\omega_{t+1}}(y_{t+1}|\omega_{t+1}) f_{\omega_{t+1}|\omega_t}(\omega_{t+1}|\omega_t) d\omega_{t+1} g(\omega_t) d\omega_t \\
&= \int \left[f_{y_{t+1}|\omega_{t+1}}(y_{t+1}|\omega_{t+1}) \int f_{\omega_{t+1}|\omega_t}(\omega_{t+1}|\omega_t) g(\omega_t) d\omega_t \right] d\omega_{t+1} \\
&= \int \left[f_{y_{t+1}|\omega_{t+1}}(y_{t+1}|\omega_{t+1}) [L_{\omega_{t+1}|\omega_t} g](\omega_{t+1}) \right] d\omega_{t+1} \\
&= [L_{y_{t+1}|\omega_{t+1}} L_{\omega_{t+1}|\omega_t} g](\omega_t).
\end{aligned}
$$

Hence:

$$
L_{y_{t+1}|\omega_t} = L_{y_{t+1}|\omega_{t+1}} L_{\omega_{t+1}|\omega_t}. \tag{39}
$$

Under stationarity, injectivity of $L_{y_t|\omega_t}$ is equivalent to injectivity of $L_{y_{t+1}|\omega_{t+1}}$, so that the Markov law of motion $f_{\omega_{t+1}|\omega_t}(\omega_{t+1}|\omega_t)$ is identified using

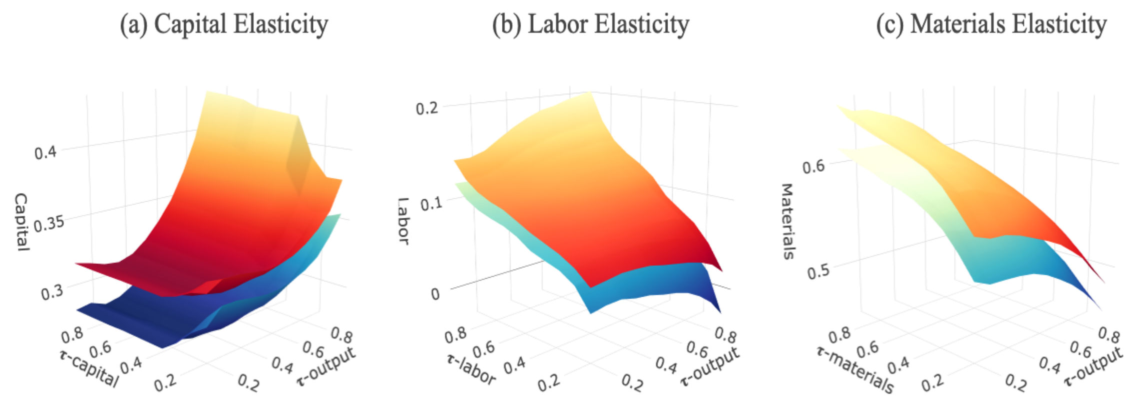
$$
L_{\omega_{t+1}|\omega_t} = L_{y_{t+1}|\omega_t} L^{-1}_{y_{t+1}|\omega_{t+1}}, \tag{40}
$$

since $f_{y_{t+1}|\omega_{t+1}}(y_{t+1}|\omega_{t+1})$ is equivalent to $f_{y_t|\omega_t}(y_t|\omega_t)$ under stationarity, $f_{\omega_{t+1}|\omega_t}(\omega_{t+1}|\omega_t)$ is identified since the densities $f_{y_t|\omega_t}(y_t|\omega_t)$ and $f_{y_{t+1}|\omega_t}(y_{t+1}|\omega_t)$ are identified from Theorem 3.1.

Proof of Corollary 3.2 :
In the absence of stationarity, the density $f_{y_{t+1}|\omega_{t+1}}$ is not the same as $f_{y_t|\omega_t}$. However, in this case, the identification strategy and result from Theorem 3.1 can be reapplied using observations $(y_{t+2}, y_{t+1}, I_{t+1})$.

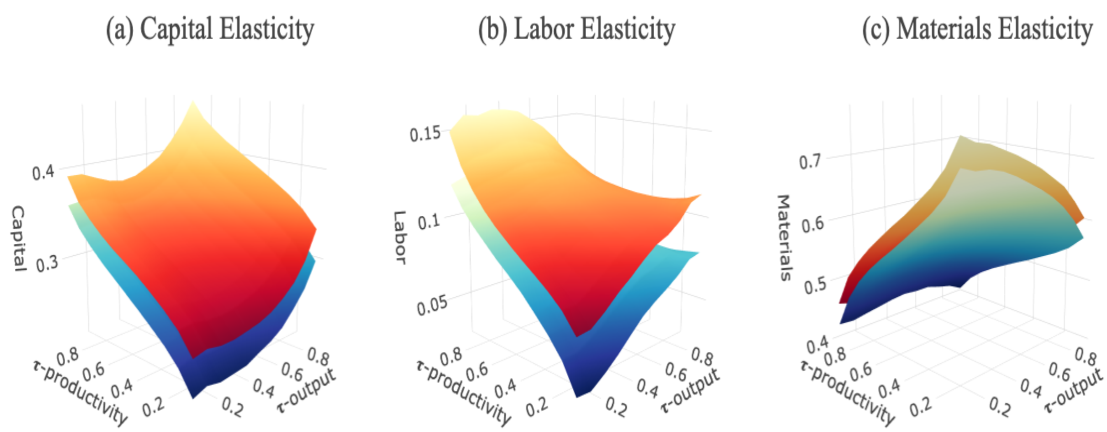
C Confidence Bands for Main Estimates

Figure 14: Output Elasticities



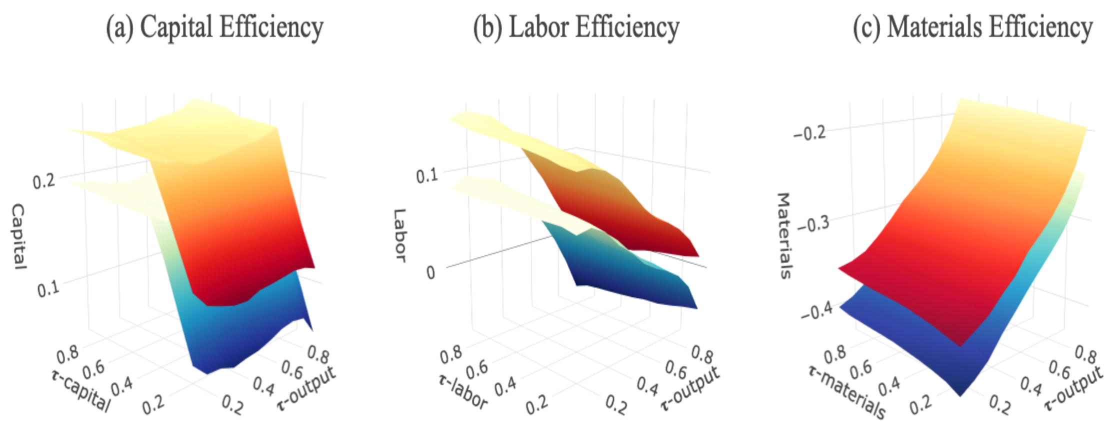
*95% Point-wise confidence bands for output elasticities. See Figure 1

Figure 15: Output Elasticities (Over Productivity)



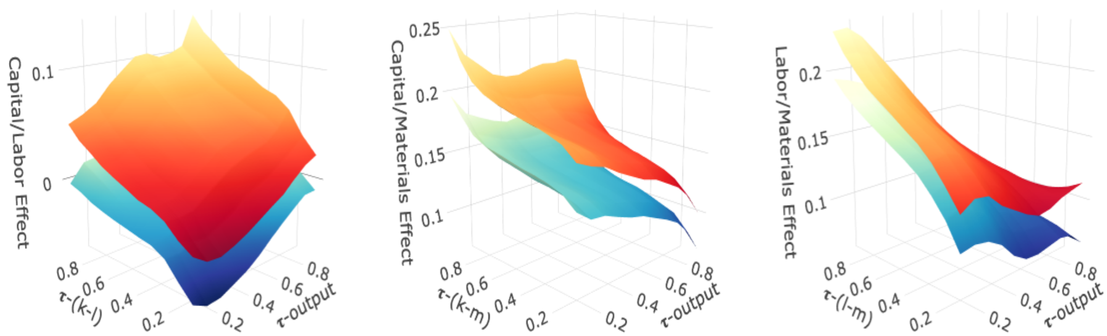
*95% Point-wise confidence bands for output elasticities (over productivity percentiles). See Figure 2

Figure 16: Effect of Productivity on Output Elasticities



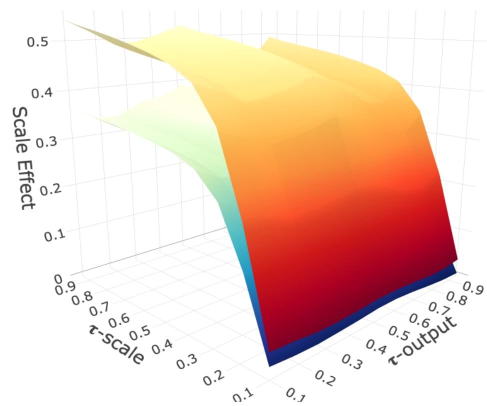
*95% Point-wise confidence bands for the marginal effects of productivity on output elasticities. See Figure 3.

Figure 17: Effect of Input Composition on Productivity



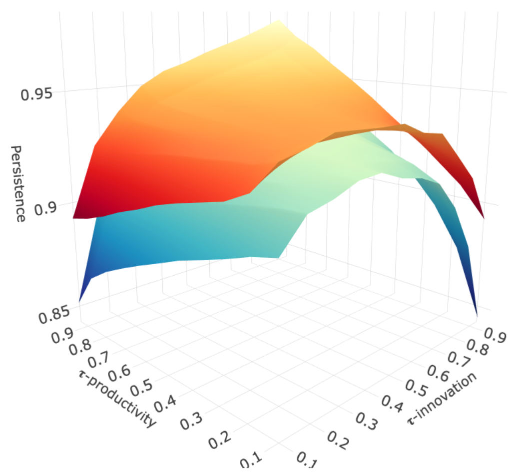
*95% Point-wise confidence bands for the input composition effects on productivity. See Figure 4.

Figure 18: Effect of Scale on Productivity



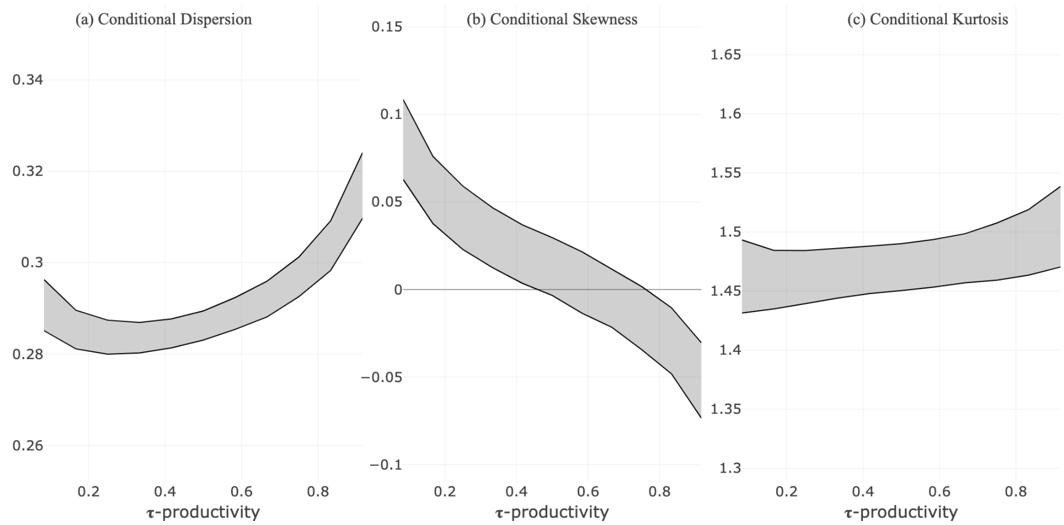
*95% Point-wise confidence bands for scale effects on productivity. See Figure 5.

Figure 19: Productivity Persistence



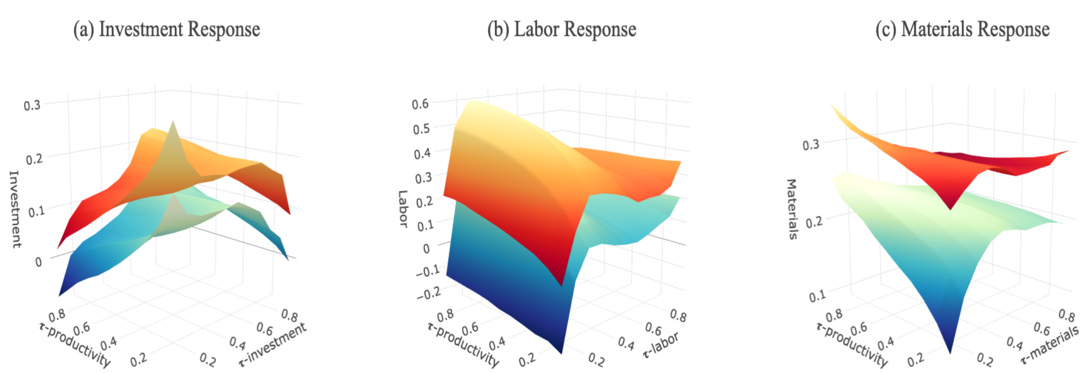
*95% Point-wise confidence bands for productivity persistence. See Figure 6.

Figure 20: Higher Moments of the Conditional Productivity Distribution



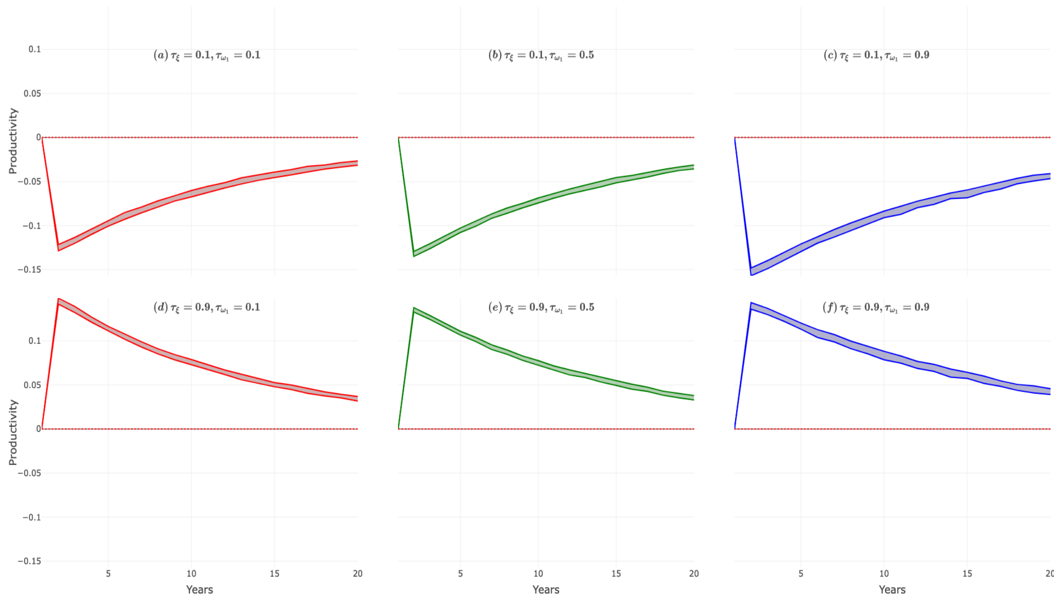
*95% Point-wise confidence bands for conditional dispersion, skewness, and kurtosis. See Figure 7.

Figure 21: Input Demand Responses to Productivity



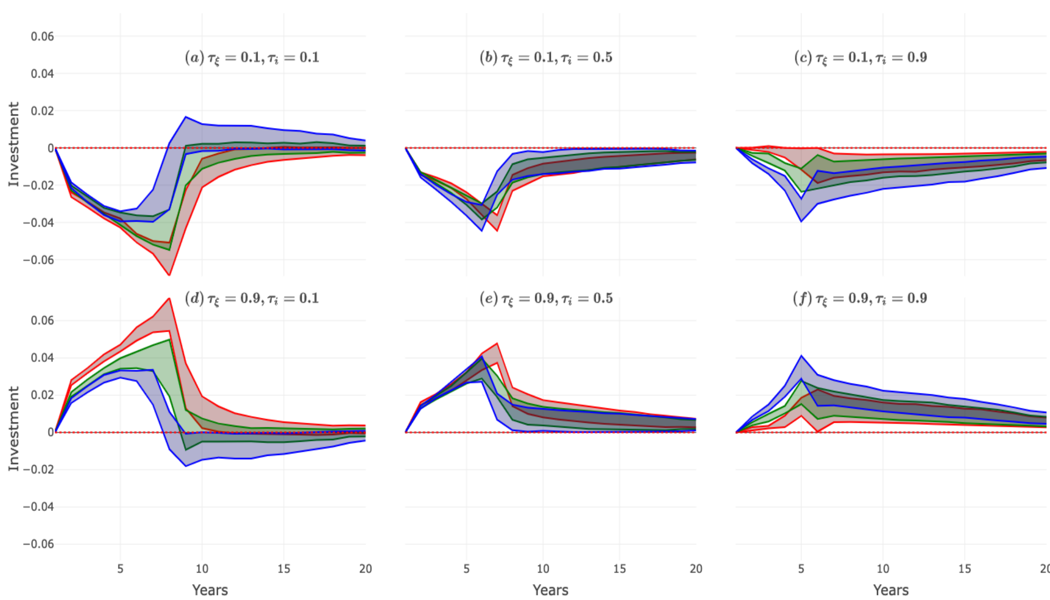
*95% Point-wise confidence bands for input demand responses to productivity. See Figure 8.

Figure 22: Impulse Response of an Innovation Shock to Productivity



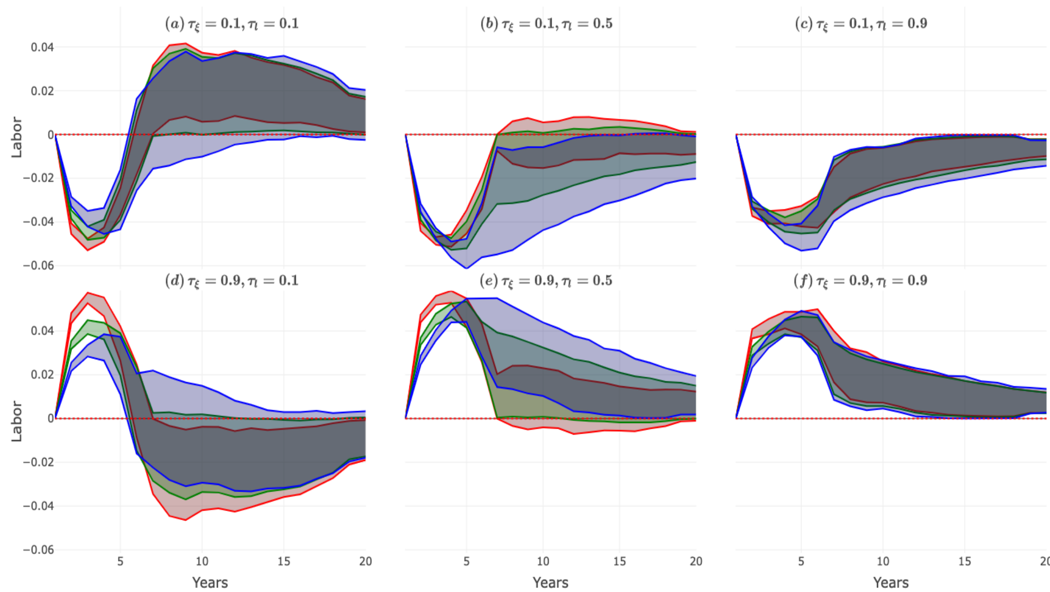
*95% Point-wise confidence bands for productivity impulse response functions. See Figure 9.

Figure 23: Impulse Response of an Innovation Shock to Investment



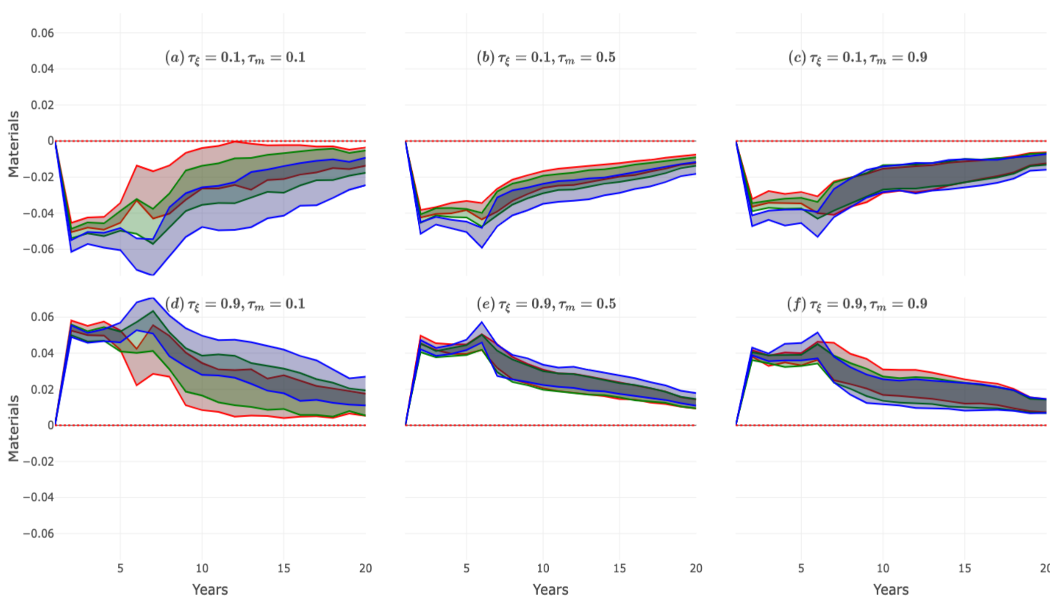
*95% Point-wise confidence bands for investment impulse response functions. See Figure 10.

Figure 24: Impulse Response of an Innovation Shock to Labor



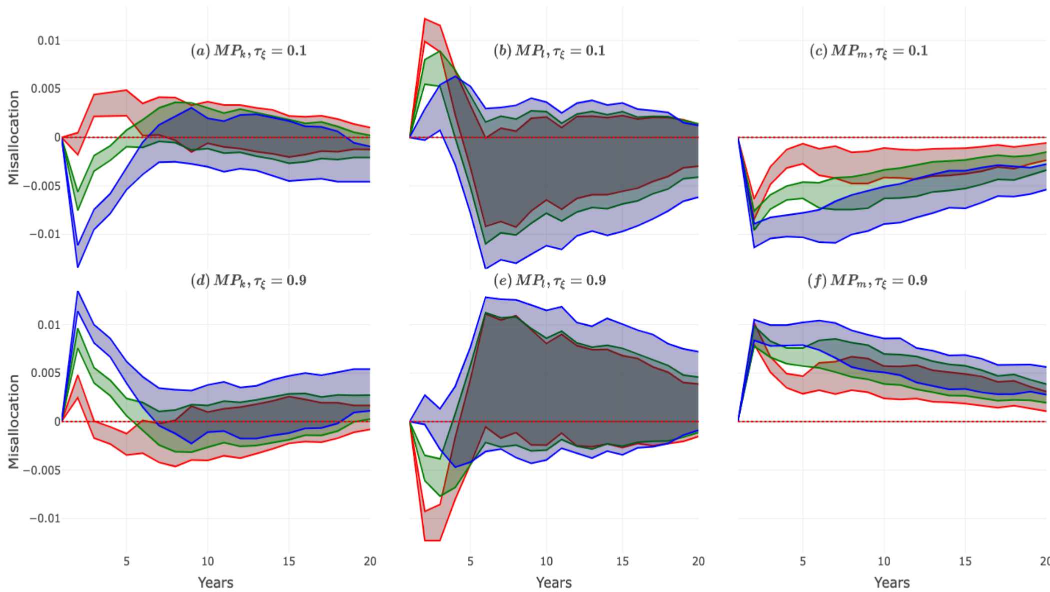
*95% Point-wise confidence bands for labor impulse response functions. See Figure 11.

Figure 25: Impulse Response of an Innovation Shock to Materials



*95% Point-wise confidence bands for materials impulse response functions. See Figure 12.

Figure 26: Impulse Response of an Innovation Shock to Misallocation



*95% Point-wise confidence bands for misallocation impulse response functions. See Figure 13.

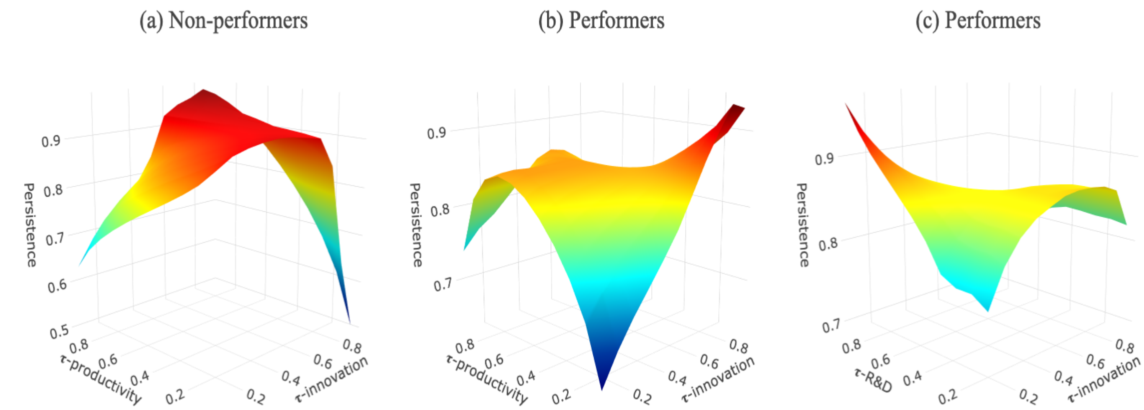
D Extensions

This section addresses the various extensions mentioned earlier in this paper. In Section D.1, I compare estimates between R&D firms and non R&D firms. I apply the estimator to study labor adjustment frictions in Section D.2. In Section D.3, I propose a correction to possible selection bias arising from non-random firm exit.

D.1 R&D Activities

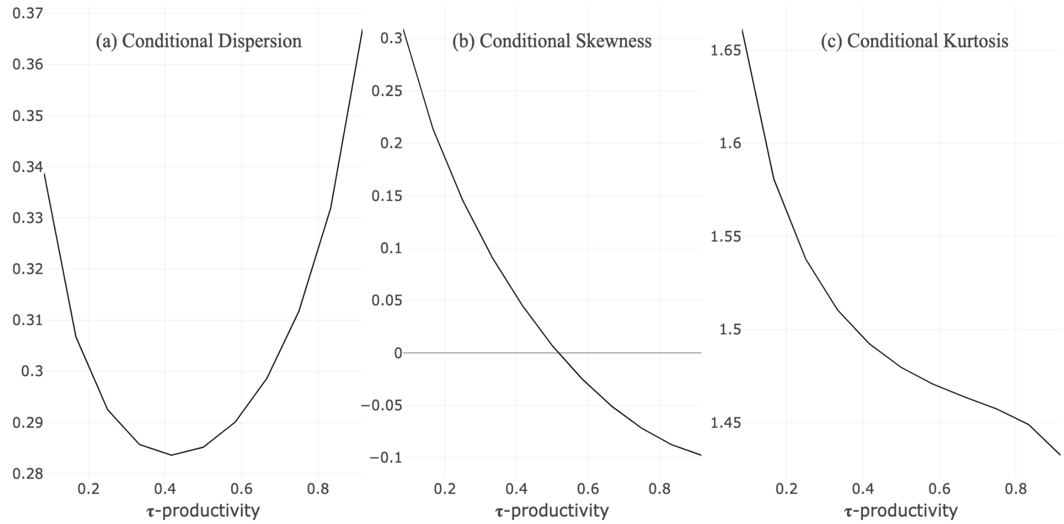
The first set of results in Figure 27 compares the estimates of productivity persistence between firms that do not perform and those that perform R&D. In panel (a), productivity persistence is plotted at fixed percentiles of previous productivity and innovation shocks for non-R&D firms. For low productivity firms, a low shock to productivity has a higher persistency (0.894) than high shocks (0.505). For high productivity firms, large innovation shocks have higher persistency (0.957) than low shocks (0.632). In panel (b), low productivity R&D firms have lower persistence than non-R&D firms hit by large negative shocks. For high productivity R&D firms, large positive productivity shock have higher persistence than non-R&D firms. Panel (c) reports persistence estimates for R&D firms evaluated at percentiles of the R&D distribution. One observation is that low R&D firms hit by large negative shocks and high R&D firms hit by large positive shocks have a lower persistence in productivity history. Overall these results suggest that R&D investment is an important factor which determines future productivity performance and that these effects vary depending on the level of R&D expenditure.

Figure 27: Productivity Persistence for Non-performing and R&D Performing Firms



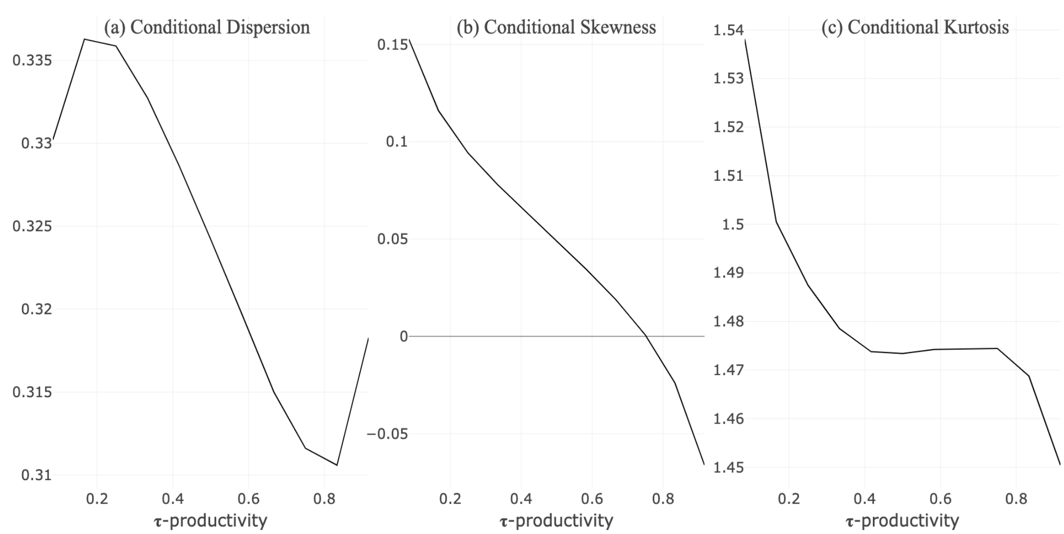
*Panel (a): Estimates of average productivity persistence for non R&D firms evaluated at τ_ξ and percentiles of previous productivity. Panel (b): Estimates of productivity persistence for R&D firms evaluated at τ_ξ and percentiles of previous productivity averaged over R&D. Panel (c): Estimates of productivity persistence for R&D firms evaluated at τ_ξ and percentiles of R&D averaged over productivity.

Figure 28: Higher Moments of the Conditional Productivity Distribution (Non R&D)



*Panel (a): Conditional dispersion evaluated at different percentiles of previous productivity. Panel (b): Conditional skewness evaluated at different percentiles of previous productivity. Panel (c): Conditional kurtosis evaluated at different percentiles of previous productivity

Figure 29: Higher Moments of the Conditional Productivity Distribution (R&D)



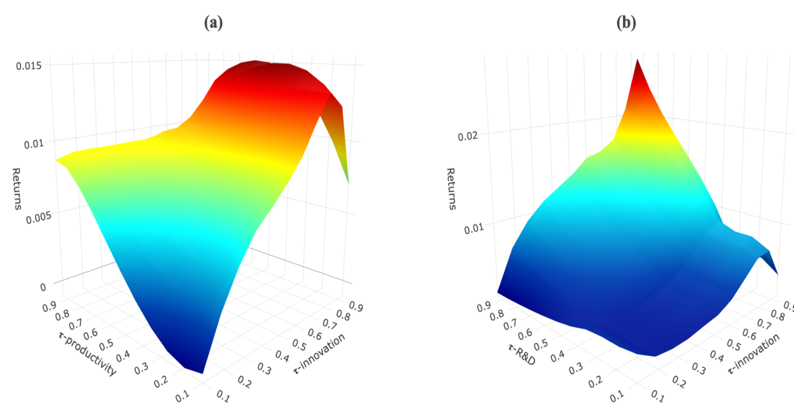
* Panel (a): Conditional dispersion evaluated at different percentiles of previous productivity. Panel (b): Conditional skewness evaluated at different percentiles of previous productivity. Panel (c): Conditional kurtosis evaluated at different percentiles of previous productivity

In Figures 28 and 29 I plot the estimates of the higher-moments of the conditional productivity distribution from the quantile estimates for non R&D and R&D performing firms.

Compared to Figure 7, non R&D firms exhibit high asymmetry in conditional dispersion. Conditional on a history of low productivity, dispersion is higher for non R&D firms than in the combined sample. A similar pattern of conditional skewness exists in this sub-sample, however the magnitude of skewness is much larger at both ends of the productivity distribution. I also find that conditional kurtosis is decreasing in productivity histories. For R&D performing firms, I find a decreasing pattern of conditional dispersion. Conditional on a history of high productivity, dispersion is low compared to firms with a low productivity history. The pattern of conditional skewness remains the same, however only low productivity firms are associated with high, positive skewness. Skewness for high productivity firms is similar as the combined sample. I also find a decreasing pattern in conditional kurtosis.

The next set of results in Figure 30 plots the returns to R&D measured by the elasticity of productivity with respect to R&D expenditures, evaluated at various percentiles of productivity, innovation shocks, and R&D. In panel (a), R&D returns are increasing in productivity levels and productivity shocks, although there is a sharp decrease in the highest productivity shock for low productivity firms. In panel (b), R&D returns are increasing in R&D expenditures and productivity shocks.

Figure 30: Returns to R&D



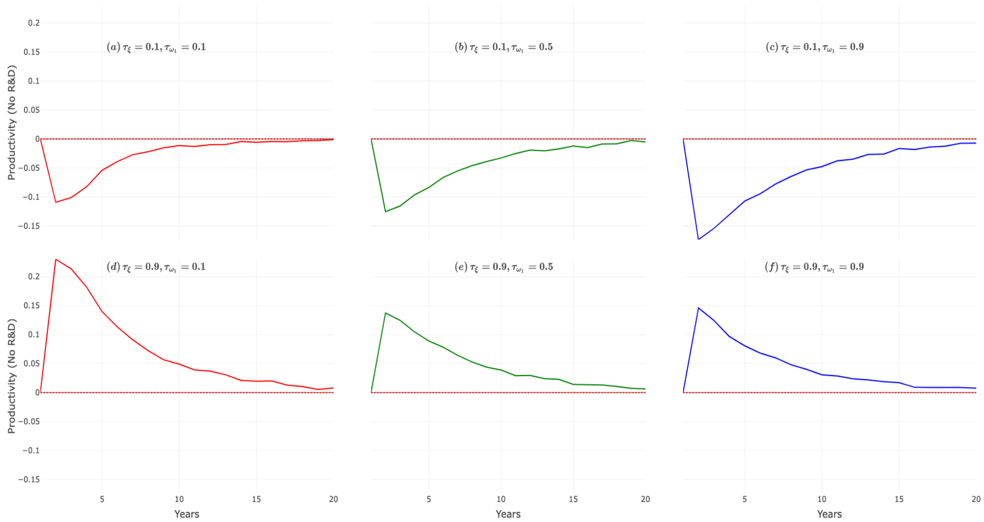
*Panel (a): Returns to R&D for firms evaluated at τ_ξ and percentiles of previous productivity averaged over R&D. Panel (b): Returns to R&D for firms evaluated at τ_ξ and percentiles of R&D averaged over productivity.

The productivity responses to innovation shocks for non R&D and R&D firms are reported in Figures 31 and 32, which shows the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and a large positive shock ($\tau_\xi = 0.9$) in panel (d-f) at various levels of initial productivity $\tau_{\omega_1} = (0.1, 0.5, 0.9)$. Following a negative productivity shock, a non R&D firm with a history of high productivity faces a 2% larger decrease in productivity than an R&D firm. Following a positive shock, a non R&D firm with a history of low productivity has a 6.5% larger increase in productivity than an R&D performing firm.

The investment responses to innovation shocks are reported in Figures 33 and 34, which shows the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and a large positive shock ($\tau_\xi = 0.9$) in panel (d-f) at various levels of investment $\tau_i = (0.1, 0.5, 0.9)$ for non R&D firms and R&D firms. The investment paths are much different compared to the paths in the combined sample in Figure 10. Overall, the adjustment of investment after a productivity shock appears to be more rapid for larger investment firms as opposed to smaller investment firms. There is distinct heterogeneity between non R&D and R&D firms. For example, following a positive productivity shock in year 7, a non R&D firm with low investment and low productivity history increases investment by 4% more than a similar R&D performing firm.

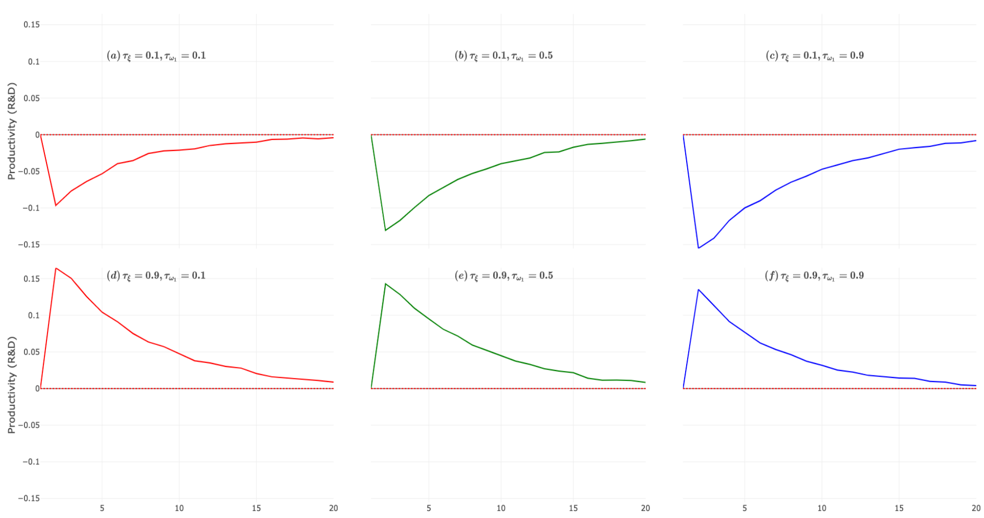
The labor responses to innovation shocks are reported in Figure 35 and 36, which shows the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and a large positive shock ($\tau_\xi = 0.9$) in panel (d-f) at various levels of labor demand $\tau_l = (0.1, 0.5, 0.9)$ for non R&D firms and R&D firms. The overall finding is that there are not significant heterogeneous labor responses between these types of firms. The materials responses to innovation shocks are reported in Figure 37 and 38, which shows the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and a large positive shock ($\tau_\xi = 0.9$) in panel (d-f) at various levels of materials demand $\tau_m = (0.1, 0.5, 0.9)$ for non R&D firms and R&D firms. Similar to the results for labor paths, the paths for materials do not appear heterogeneous for the two types of firms.

Figure 31: Impulse Response of an Innovation Shock to Productivity (Non R&D Firms)



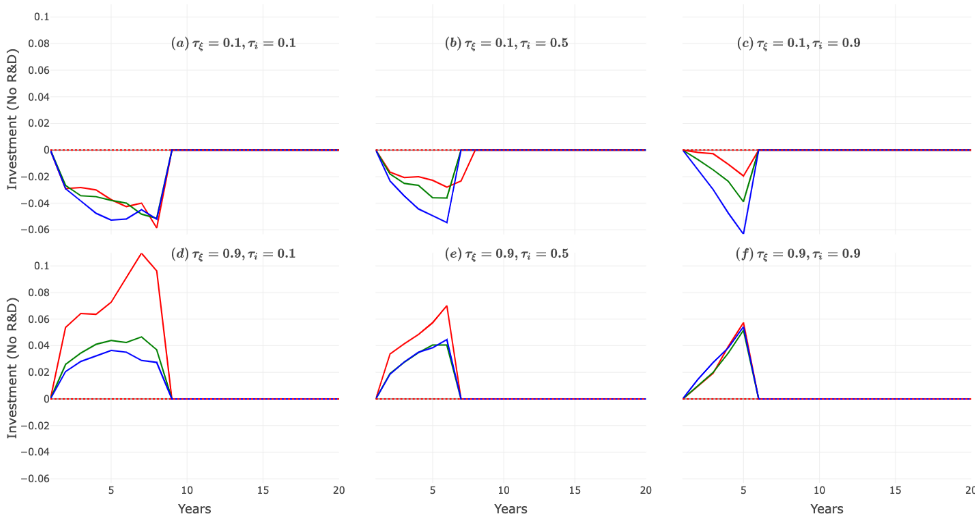
*Top row: Differences in productivity between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in productivity between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity.

Figure 32: Impulse Response of an Innovation Shock to Productivity (R&D Firms)



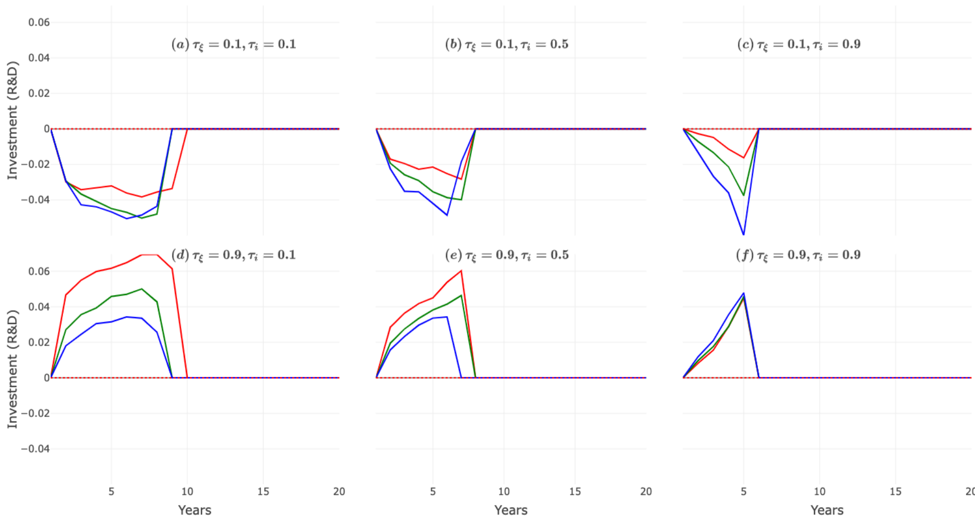
*Top row: Differences in productivity between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in productivity between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity.

Figure 33: Impulse Response of an Innovation Shock to Investment (Non R&D Firms)



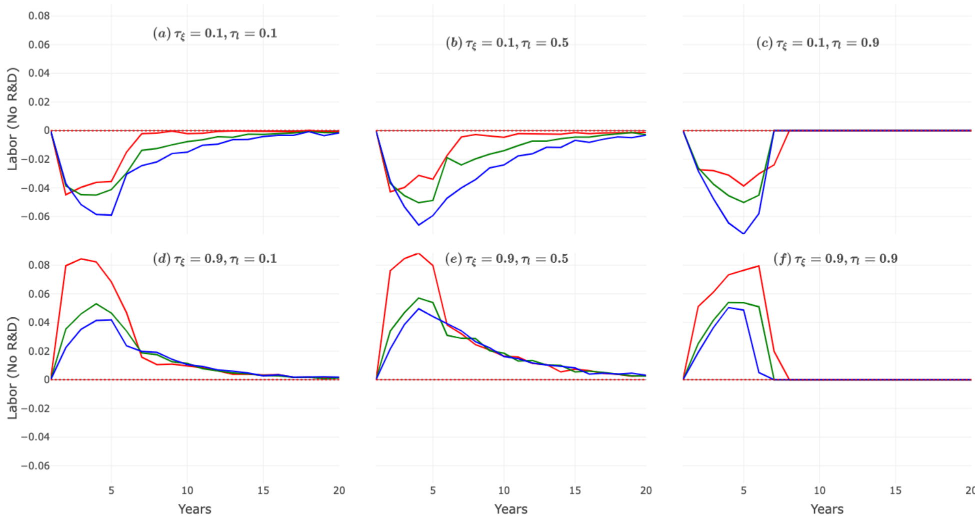
*Top row: Differences in investment between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in investment between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 34: Impulse Response of an Innovation Shock to Investment (R&D Firms)



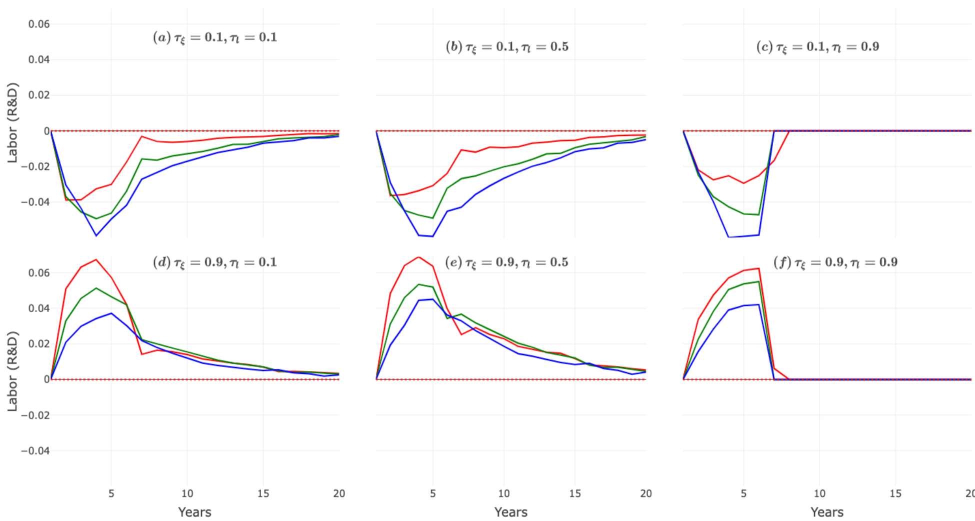
*Top row: Differences in investment between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in investment between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 35: Impulse Response of an Innovation Shock to Labor (Non R&D Firms)



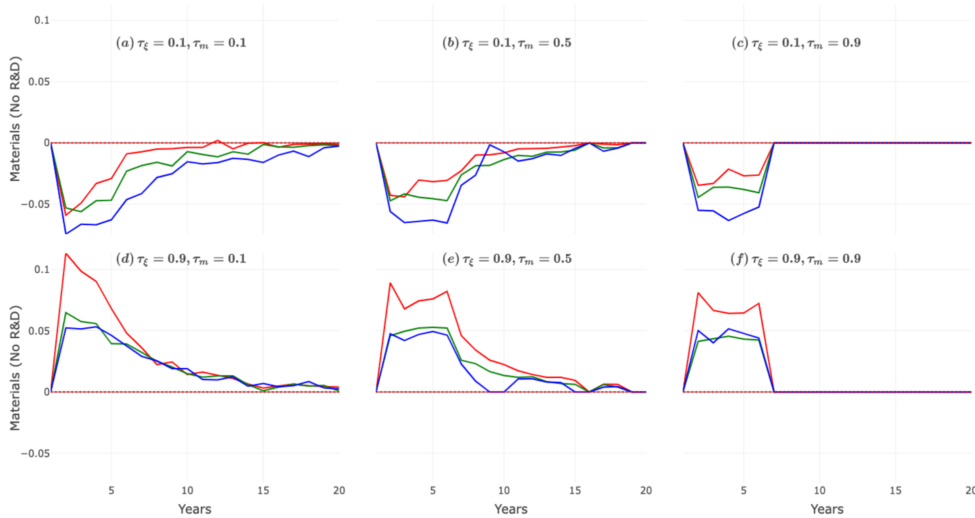
*Top row: Differences in labor between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in labor between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 36: Impulse Response of an Innovation Shock to Labor (R&D Firms)



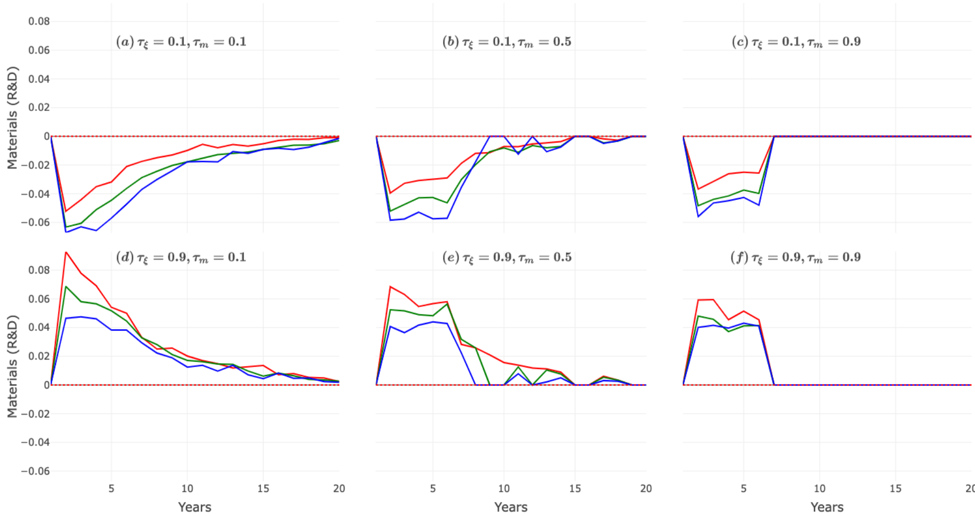
*Top row: Differences in labor between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in labor between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 37: Impulse Response of an Innovation Shock to Materials (Non R&D Firms)



*Top row: Differences in materials between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in materials between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 38: Impulse Response of an Innovation Shock to Materials (R&D Firms)

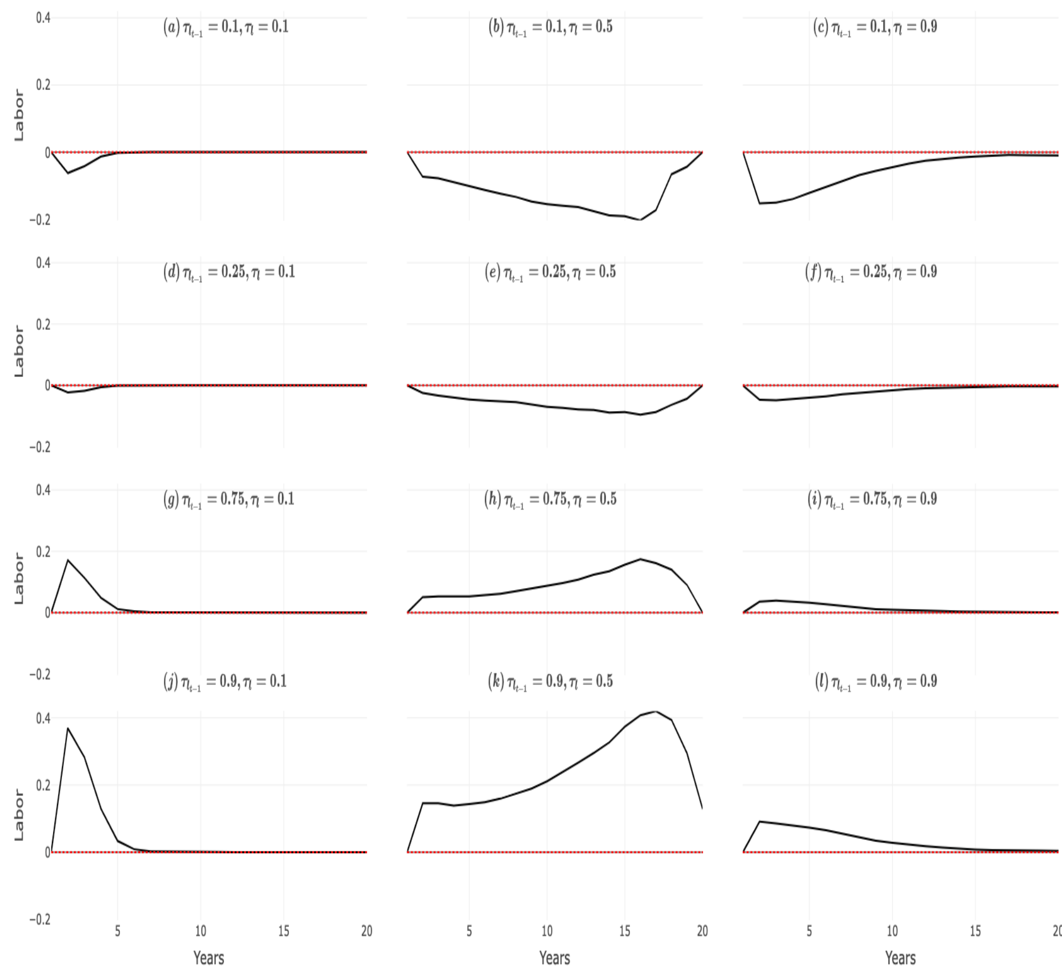


*Top row: Differences in materials between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in materials between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

D.2 Labor Dynamics

This section extends the labor demand function to include lagged labor from Equation (7). In Figure 39, I report impulse response function for firms of different labor demand size who are hit with differently sized shocks to previous labor. The results show heterogeneous responses for high labor demand firms hit by low shocks and low labor demand firms hit by high shocks.

Figure 39: Impulse Response of Adjustment Shocks to Labor



*Top row: Difference between firms hit with low labor shock $\tau_{l_{t-1}} = 0.1$ and medium shock $\tau_{l_{t-1}} = 0.5$ at different levels of labor demand. Second row: Difference between firms hit with labor shock $\tau_{l_{t-1}} = 0.25$ and medium shock $\tau_{l_{t-1}} = 0.5$ at different levels of labor demand. Third row: Difference between firms hit with labor shock $\tau_{l_{t-1}} = 0.75$ and medium shock $\tau_{l_{t-1}} = 0.5$ at different levels of labor demand. Bottom row: Difference between firms hit with high labor shock $\tau_{l_{t-1}} = 0.9$ and medium shock $\tau_{l_{t-1}} = 0.5$ at different levels of labor demand. Labor demand is evaluated at percentiles of lagged labor averaged over capital and productivity.

D.3 Correcting for Selection Bias

The estimation procedure presented here can be adapted to correct for non-random firm exit in the framework of Olley and Pakes (1996) and Dermirer (2020). An exit rule is part of a Markov perfect Nash equilibrium, which determines a threshold level of productivity for which firms will stay in operation. The decision to stay in operation or exit is given by:

$$\chi_{it} = \begin{cases} 1 & \text{if} \quad \omega_{it} \geq \underline{\omega}_t(k_{it}) \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

The productivity threshold is determined by a firm's current capital stock. Firms with larger capital stocks can expect larger future returns for any given level of current productivity. Using the specification for the productivity process in Equation (20), the decision to stay in operation can be written as:

$$\begin{aligned} Q_t^{\omega}(\omega_{it-1}, \xi_{it}) \geq \underline{\omega}_t(k_{it}), & \\ \xi_{it} \geq Q_t^{\omega^{-1}}(\omega_{it-1}, \underline{\omega}_t(k_{it})), & \\ \xi_{it} \geq Q_t^{\omega^{-1}}(\omega_{it-1}, k_{it}), & \\ \xi_{it} \geq \underline{\omega}_t(\omega_{it-1}, k_{it}), & \end{aligned} \tag{42}$$

where the second inequality follows from the monotonicity restriction in Assumption 2.2. Provided that the Markov process for productivity is exogenous $\Pr(\omega_{it}|\omega_{it-1}, \mathcal{I}_{it-1}) = \Pr(\omega_{it}|\omega_{it-1})$, the innovation shocks to productivity will be independent of current capital stock since $k_{it} \in \mathcal{I}_{it-1}$. This allows me to characterize the conditional distribution of innovation shocks as

$$\xi_{it}|(k_{it}, \omega_{it-1}) \sim U(0, 1).$$

The cutoff for which firms stay in operation can written as

$$\underline{\omega}_t(\omega_{it-1}, k_{it}) = \text{Prob}(\chi_{it} = 1|\omega_{it-1}, k_{it}) \equiv p(\omega_{it-1}, k_{it}). \tag{43}$$

Therefore, firms that receive an innovation shock greater than $p(\omega_{it-1}, k_{it})$ will continue to operate. The distribution of productivity innovations conditional on (k_{it}, ω_{it-1}) and $\chi_{it} = 1$ is

$$\xi_{it}|(k_{it}, \omega_{it-1}, \chi_{it} = 1) \sim U(p(\omega_{it-1}, k_{it}), 1). \tag{44}$$

To see how this could be used to correct for selection bias, consider a simple linear random coefficient model for productivity: $\omega_{it} = \rho(\xi_{it})\omega_{it-1}$. The independence assumptions imply:

$$\begin{aligned}\text{Prob}(\omega_{it} \le \rho(\tau)\omega_{it-1}|\omega_{it-1}, k_{it}, \chi_{it} = 1) \\ &= \text{Prob}(\xi_{it} \le \tau|\omega_{it-1}, k_{it}, \chi_{it} = 1) \\ &= \frac{\tau - p(\omega_{it-1}, k_{it})}{1 - p(\omega_{it-1}, k_{it})} \equiv G(\tau, p).\end{aligned} \tag{45}$$

This implies that for a current draw of productivity $\omega_{it}^{(m)}$, the persistence parameter, $\rho(\tau)$, can be estimated using the *rotated* quantile regression:

$$\hat{\rho}(\tau_q)^{(s+1)} = \underset{\rho(\tau_q)}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{m=1}^{M} \chi_{it}\Big[G(\tau_q, p)(\omega_{it}^{(m)} - \rho\omega_{it-1}^{(m)})^{+} + (1 - G(\tau_q, p))(\omega_{it}^{(m)} - \rho\omega_{it-1}^{(m)})^{-}\Big], \tag{46}$$

where $a^{+} = max(a, 0)$, $a^{-} = max(-a, 0)$, and $p(\omega_{it-1}, k_{it})$ can be estimated from a probit regression on $\omega_{it-1}^{(m)}$ and k_{it}. This estimator is similar to the one proposed by Arellano and Bonhomme (2017), although in my case the shift in the productivity rank is easier to characterize from the structural model used here. Implementing this selection correction is straight-forward in standard quantile regression packages. For example, in quantreg for R, this requires using an individual-specific τ in the dual equality constraints. The estimator uses the Frisch-Newton linear programming algorithm in Portnoy and Koenker (1997), which can be implemented using **rq.fit.fnb**. The consequence of selection bias in this setting, is the entire process of $\rho(\tau)$ may be biased. The amount of bias is likely to be larger at the bottom of the productivity distribution, where the probability of exit is higher. Therefore, I must also control for selection bias for $\tau \le \tau_1$ and $\tau > \tau_Q$ in the original model. I do this by adopting a control function approach in the tails. To illustrate, I use the simple AR(1) model for productivity at $\tau \le \tau_1$:

$$\omega_{it} = \rho(\tau_1)\omega_{it-1} + v_{it} + u_{it}, \quad \omega_{it} \le \rho(\tau_1)\omega_{it-1}, \tag{47}$$

where v_{it} denotes the unobservable component of productivity that is correlated to the firm's exit decision, and u_{it} denotes an i.i.d. shock that is assumed to be exponentially distributed. The issue of selection arises because

$$\mathbb{E}[\omega_{it}|\omega_{it-1}, \chi_{it} = 1] = \rho(\tau_1)\omega_{it-1} + \mathbb{E}[v_{it}|\omega_{it-1}, \chi_{it} = 1]. \tag{48}$$

Note that $\mathbb{E}[v_{it}|\omega_{it-1}, \chi_{it} = 1] \neq 0$ causes selection bias for productivity estimates at $\tau \leq \tau_1$. Provided that the density of ω_{it} conditional on ω_{it-1} is positive in a region about $\underline{\omega}_{it}$, following Olley and Pakes (1996), I invert the selection equation as a function of the propensity score $p = p(\omega_{it-1}, k_{it})$ and ω_{it-1}. Therefore, I have the following equation:

$$\mathbb{E}[\omega_{it}|\omega_{it-1}, \chi_{it} = 1] = \rho(\tau_1)\omega_{it-1} + s_1(p, \omega_{it-1}), \tag{49}$$

where $s_1(\cdot)$ denotes the sample selection correction function. I approximate this function by a second degree polynomial in p and ω_{it-1}. Then, an estimate for the exponential parameter is updated from

$$\hat{\lambda}_\rho^{-(s)} = \frac{-\sum_{n=1}^{N}\sum_{t=2}^{T}\sum_{m=1}^{M}\mathbb{1}\{\omega_t^{(m)} \leq \hat{\rho}(\tau_1)^{(s)}\omega_{t-1}^{(m)} + \hat{s}_1(p_t, \omega_{t-1}^{(m)})\}}{\sum_{n=1}^{N}\sum_{t=2}^{T}\sum_{m=1}^{M}(\omega_t^{(m)} - \hat{\rho}(\tau_1)^{(s)}\omega_{t-1}^{(m)} - \hat{s}_1(p_t, \omega_{t-1}^{(m)}))\mathbb{1}\{\omega_t^{(m)} \leq \hat{\rho}(\tau_1)^{(s)}\omega_{t-1}^{(m)} + \hat{s}_1(p_t, \omega_{t-1}^{(m)})\}}. \tag{50}$$

The algorithm proceeds similarly as before. Given an initial parameter value $\hat{\theta}^0$, iterate on $s = 0, 1, 2, \ldots,$ in the following two-step procedure until convergence to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$\begin{aligned} g_i(\omega_i^T; \hat{\theta}^{(s)}) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T, \chi_i^T; \hat{\theta}^{(s)}) \propto \\ \prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}, \chi_i^T; \hat{\beta}^{(s)}) f(l_{it}|k_{it}, \omega_{it}, \chi_i^T; \hat{\alpha}_l{}^{(s)}) f(m_{it}|k_{it}, l_{it}, \omega_{it}, \chi_i^T; \hat{\alpha}_m^{(s)}) \\ \times f(i_{it}|k_{it}, \omega_{it}, \chi_i^T; \hat{\delta}^{(s)}) \prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1}, \chi_i^T; \hat{\rho}^{(s)}) p(k_{it}, \omega_{it-1}; \hat{\rho}_\chi^{(s)}) f(\omega_{i1}|k_{i1}; \hat{\rho}_{\omega_1}^{(s)}). \end{aligned}$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve

$$\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}^{(m)}; \beta(\tau_q)) \right),$$

$$\hat{\boldsymbol{\alpha}}_l(\tau_q)^{(s+1)} = \underset{\boldsymbol{\alpha}_\ell(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(l_{it} - \sum_{j=1}^{J_\ell} \alpha_{\ell,j}(\tau_q) \phi_{l,j}(k_{it}, \omega_{it}^{(m)}) \right),$$

$$\hat{\boldsymbol{\alpha}}_m(\tau_q)^{(s+1)} = \underset{\boldsymbol{\alpha}_m(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(m_{it} - \sum_{j=1}^{J_m} \alpha_{m,j}(\tau_q) \phi_{m,j}(k_{it}, l_{it}, \omega_{it}^{(m)}) \right),$$

$$\hat{\boldsymbol{\delta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\delta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(i_{it} - \sum_{j=1}^{J_\iota} \delta_j(\tau_q) \phi_{\iota,j}(k_{it}, \omega_{it}^{(m)}) \right),$$

$$\hat{\boldsymbol{\rho}}_\chi^{(s+1)} = \underset{\boldsymbol{\rho}_\chi}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \left(\chi_{it} \ln \Phi(w(k_{it}, \omega_{it-1}^{(m)}; \rho_\chi)) + (1 - \chi_{it}) \ln(1 - \Phi(w(k_{it}, \omega_{it-1}^{(m)}; \rho_\chi))) \right),$$

$$\hat{\boldsymbol{\rho}}(\tau_q)^{(s+1)} = \underset{\rho(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \chi_{it} \left(G(\tau_q, \hat{p})(\omega_{it}^{(m)} - \rho\omega_{it-1}^{(m)})^+ + (1 - G(\tau_q, \hat{p}))(\omega_{it}^{(m)} - \rho\omega_{it-1}^{(m)})^- \right),$$

$$\hat{\boldsymbol{\rho}}_{\omega_1}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\rho}_{\omega_1}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{m=1}^{M} \psi_{\tau_q} \left(\omega_{i1}^{(m)} - \sum_{j=1}^{J_{\omega_1}} \rho_{\omega_1}(\tau_q) \phi_{\omega_1,j}(k_{i1}) \right),$$

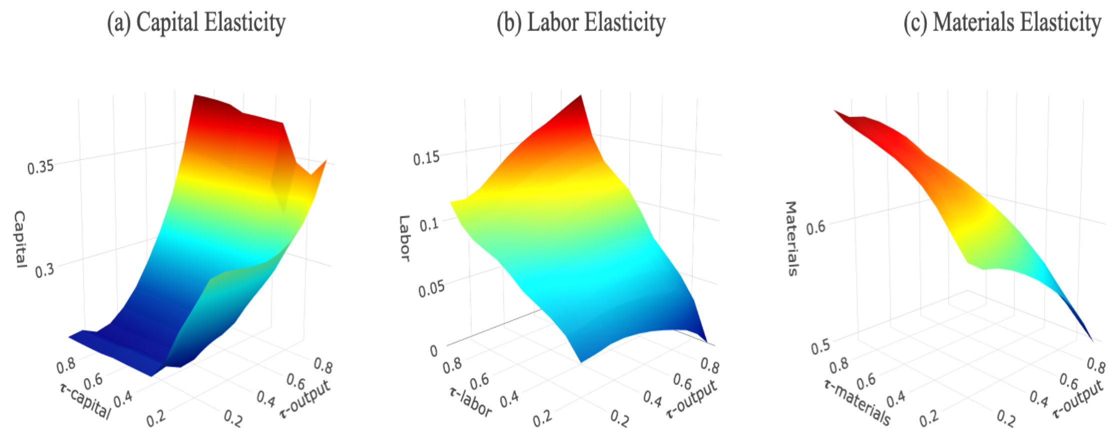
where $\psi_\tau(u) = (\tau - \mathbb{1}\{u < 0\})u$ is the "check" function from quantile regression. Here, ρ_χ are the parameters estimated from a probit regression of the exit decision on capital and lagged productivity from the third-to-last equation in the above M-step procedure. I approximate the function $w(k_{it}, \omega_{it-1}; \rho_\chi)$ by a second-order polynomial in k_{it} and ω_{it-1}. Kernel density estimators can also be employed to estimate the selection probabilities. The exponential parameter for $\tau \leq \tau_1$ is updated using Equation (50) and for $\tau > \tau_Q$:

$$\hat{\lambda}_\rho^{+(s)} = \frac{\sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \mathbb{1}\{\omega_t^{(m)} > \hat{\rho}(\tau_Q)^{(s)}\omega_{t-1}^{(m)} + \hat{s}_2(p_t, \omega_{t-1}^{(m)})\}}{\sum_{n=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} (\omega_t^{(m)} - \hat{\rho}(\tau_Q)^{(s)}\omega_{t-1}^{(m)} - \hat{s}_2(p_t, \omega_{t-1}^{(m)}))\mathbb{1}\{\omega_t^{(m)} > \hat{\rho}(\tau_Q)^{(s)}\omega_{t-1}^{(m)} + \hat{s}_2(p_t, \omega_{t-1}^{(m)})\}}, \tag{51}$$

where $\hat{s}_2(\cdot)$ denotes another sample selection correction function. Selection correction methods for nonseparable quantile models are studied by Arellano and Bonhomme (2017), but to my knowledge, has not been applied to non-linear panel data models. This extension may provide a useful starting point for combining the two literatures.

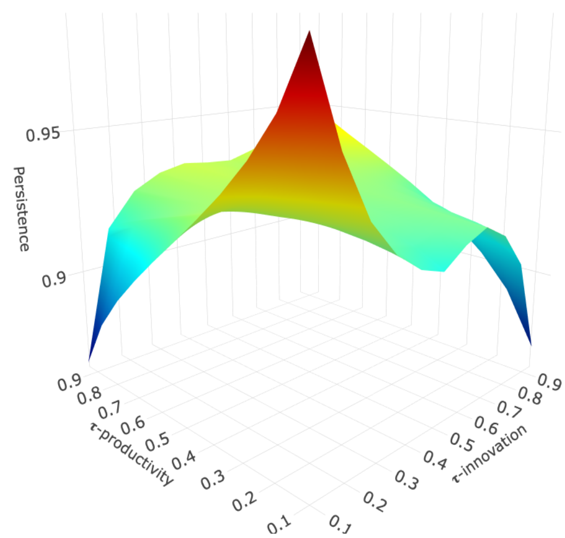
To examine the extent of selection bias, I re-estimate the original model with the proposed correction. Figure 40 reports the estimates of the average capital elasticity corrected for selection. Compared to the uncorrected estimates from Figure 1, the corrected elasticities are uniformly smaller. This makes it difficult to support that selection leads to a negative bias in the capital coefficient as argued by OP. The estimates for labor and materials are almost identical.

Figure 40: Output Elasticities (Selection Corrected)



*Panel (a): Capital elasticity evaluated at τ_η and percentiles of capital τ_k averaged over values of ω_{it} and (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor elasticity evaluated at τ_η and percentiles of labor τ_l averaged over values of ω_{it} and (k_{it}, m_{it}). Panel (c): Materials elasticity evaluated at τ_η and percentiles of materials τ_m averaged over values of ω_{it} and (k_{it}, l_{it}).

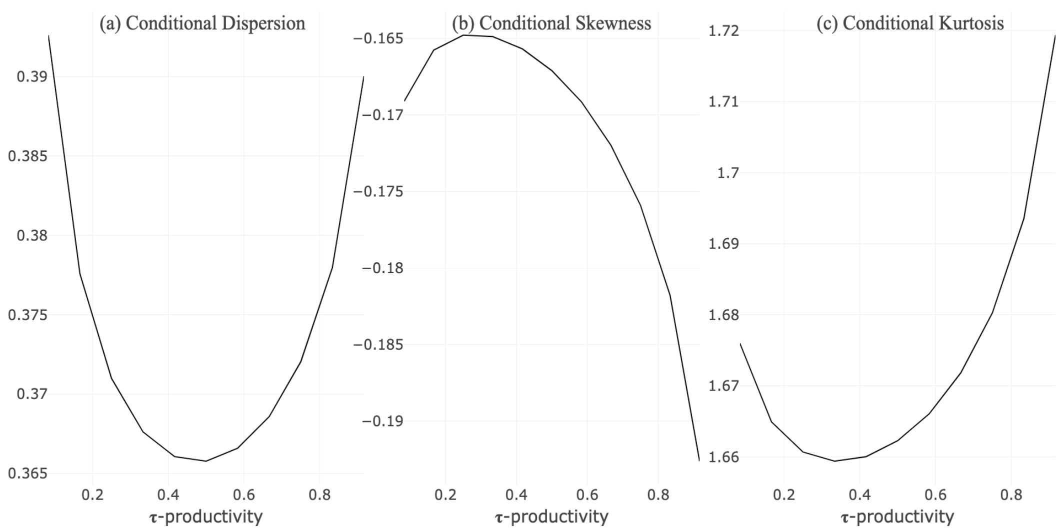
Figure 41: Productivity Persistence (Selection Corrected)



*Estimates of average productivity persistence evaluated at τ_ξ and percentiles of previous productivity.

Figure 41 plots the selection corrected productivity persistence over percentiles of innovation shocks and productivity histories. The difference between these estimates and the uncorrected estimates in Figure 6 reveals a higher estimate of persistence associated with firms with histories of low productivity subject to a negative productivity shock. This result is intuitive, as these types of firms have a higher probability of exiting the market.

Figure 42: Higher Moments of the Conditional Productivity Distribution (Selection Corrected)

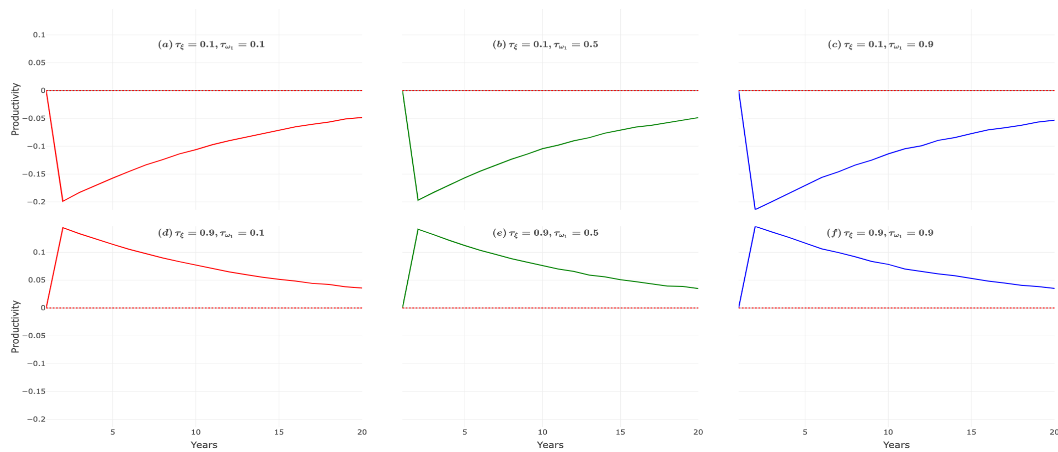


*Panel (a): Conditional dispersion evaluated at different percentiles of previous productivity. Panel (b): Conditional skewness evaluated at different percentiles of previous productivity. Panel (c): Conditional kurtosis evaluated at different percentiles of previous productivity

In Figure 42, I break down the characteristics of the conditional distribution using the estimates from the selection corrected productivity process. The results in panel (a) show a higher degree of asymmetry in uncertainty than Figure 7. Selection corrected estimates reveal that dispersion is high for both firms with low histories of productivity and high histories. Additionally, panel (b) reveals that skewness is negative, conditional on any rank of productivity histories. Lastly, estimates of conditional kurtosis share a similar pattern as the uncorrected estimates, although the selection corrected estimates are slightly higher.

The productivity responses to innovation shocks are reported in Figure 43, which show the impact of a large negative shock ($\tau_\xi = 0.1$) in panel (a-c) and a large positive shock ($\tau_\xi = 0.9$) in panel (d-f) for various levels of initial productivity $\tau_{\omega_1} = (0.1, 0.5, 0.9)$. For firms with the lowest initial productivity, a large negative innovation shock decreases productivity by 19.9%, while a large positive shock increases productivity by 14.4%. For firms with the highest initial productivity, a large negative innovation shock decreases productivity by 21.4% and a large positive shock increases productivity by about 14.7%. The estimates correcting for selection have the most pronounced differences when firms are hit by low productivity shocks, as more firms are likely to exit at this level. After correcting for selection, productivity decreases by a larger amount. This is because conditional on staying in the market, industry productivity tends to appear higher than the unobserved distribution of productivity because of firm exits at low productivity realizations. That is, there is positive selection into staying in operation.

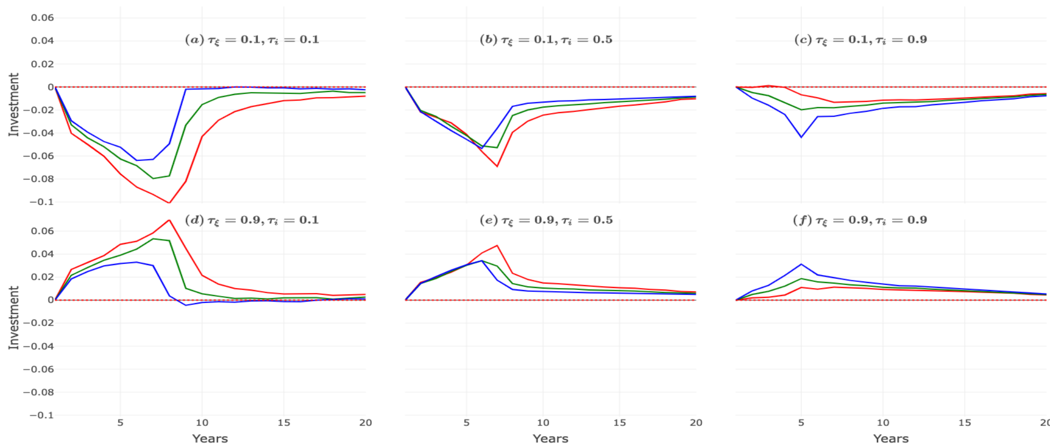
Figure 43: Impulse Response of an Innovation Shock to Productivity
(Selection Corrected)



*Top row: Differences in productivity between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences in productivity between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity.

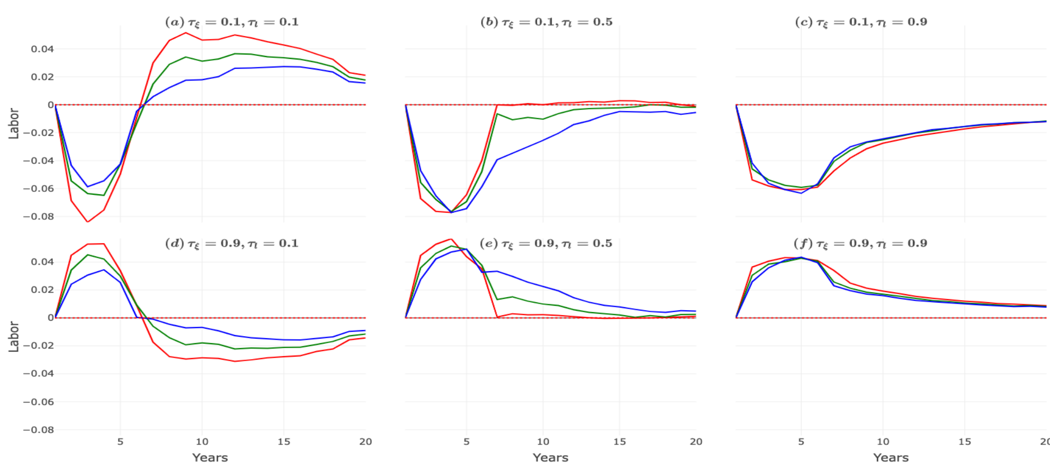
The investment responses to innovation shocks are reported in Figure 44, which shows the impact of a negative productivity shock in panel (a-c) and a positive productivity shock in panel (d-f) for various levels of investment demand $\tau_i = (0.1, 0.5, 0.9)$ and initial productivities. The main finding is that for low and medium investment levels, firms with histories of low productivity have a larger drop in investment following a bad productivity shock after correcting for selection.

Figure 44: Impulse Response of an Innovation Shock to Investment
(Selection Corrected)



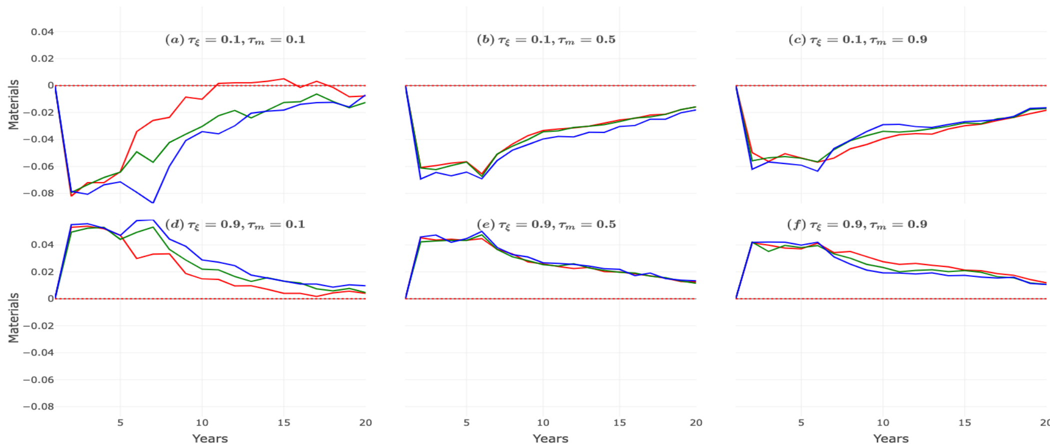
*Top row: Differences in capital between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand. Bottom row: Differences in capital between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 45: Impulse Response of an Innovation Shock to Labor
(Selection Corrected)



*Top row: Differences in labor between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand. Bottom row: Differences in labor between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

Figure 46: Impulse Response of an Innovation Shock to Materials
(Selection Corrected)



*Top row: Differences in materials between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand. Bottom row: Differences in materials between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand. Low, medium, and high initial productivity paths are denoted by the red, green, and blue lines respectively.

The labor responses to innovation shocks are reported in Figure 45, which shows the impact of a negative productivity shock in panel (a-c) and a positive productivity shock in panel (d-f) for various levels of labor demand $\tau_l = (0.1, 0.5, 0.9)$ and initial productivities. In Figure 46, I plot similar paths for materials. After correcting for selection, there is a larger drop in demand for labor and materials following a negative productivity shock.

In conclusion, the extensions in the Appendix suggest that heterogeneity can be pronounced due to labor adjustment frictions, R&D performance, and correcting for econometric issues such as selection bias.